Aix-Marseille Université
Laboratoire d'Analyse Topologie Probabilités
ÉCOLE DOCTORALE DE MATHÉMATIQUES ET INFORMATIQUE
E.D. numéro 184

THÈSE

Présentée pour obtenir le grade de Docteur
Discipline : Mathématiques

par

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Titre :

Analyse de modèles en mécanique des fluides compressibles

Soutenue publiquement le 15 Décembre 2012

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Chapitre 1

Introduction

La dynamique des fluides est la branche de la physique qui traite les mouvements des fluides, qu'ils soient liquides ou gaz. Elle englobe ainsi tous les phénomènes d'écoulement qui se présente quotidiennement dans notre environnement immédiat, qu'il s'agisse de l'écoulement de l'air ou de l'eau dans un environnement naturel ou fabriqué par l'homme. Cette désignation de mécanique des fluides est souvent remplacée par d'autres appellations correspondant au type du fluide, comme l'hydrodynamique pour l'écoulement de l'eau, l'aérodynamique pour l'air, la magnétohydrodynamique pour les plasmas...etc.

La résolution d'un problème de dynamique des fluides nécessite l'élaboration d'un modèle mathématique permettant le calcul de diverses quantités comme la vitesse, la pression, la densité et la température en tant que fonctions de l'espace et du temps. Un modèle fréquemment utilisé est celui donné par les équations de Navier-Stokes compressible ou incompressible. La présence irrésistible de la mécanique des fluides dans la nature qui nous entoure, motive la nécessité de l'étude de ces équations. Au début des années trente du 20^{e} siècle, J.Leray a publié ses célèbres travaux [16], [17], [18], en particulier [18] sur les équations de Navier-Stokes incompressible. Ces résultats ont été le point de départ de très nombreux travaux de recherche dans le cadre des mathématiques actuelles, en particulier les travaux récents de P.L.Lions [14] et E.Feireisl [8] qui ont permis une avancé considérable pour la compréhension des équations de Navier-Stokes compressible.

Dans notre travail, on s'intéresse aux équations de Stokes compressibles qui sont obtenues à partir des équations de Navier-Stokes compressible en négligeant certains termes dans l'équation de quantité de mouvement. En particulier on démontre l'existence de solution du problème de Stokes avec une loi d'état générale. Cette existence est démontrée de deux manières différrentes, une première méthode consiste à passer à la limite sur des solutions approchées obtenues avec une régularisation par viscosité de l'équation de conservation de la masse. Une deuxième méthode consiste à passer à la limite sur des solutions approchées obtenues avec un schéma numérique discrétisant les équations. Ce schéma combine la méthode des éléments finis et la méthode des volumes finis. La discrétisation proposée est très proche de celle donnée dans [19] pour la solution des équations de Navier-Stokes barotropes et dans [20] pour des écoulements diphasiques. Cette deuxième méthode est particulièrement intéressante car elle donne la convergence d'un schéma numérique couramment utilisé à l'Institut de Radioprotection et de Sûreté Nucléaire (IRSN).

L'IRSN réalise des recherches, des expertises et des travaux dans les domaines de la sûreté nucléaire, de la protection contre les rayonnements ionisants, du contrôle et de la protection des matières nucléaires, et de la protection contre les actes de malveillance. Une partie essentielle de l'analyse de sûreté consiste à étudier des accidents graves. les écoulements intéressants dans ce sens sont compressibles. Il est donc nécessaire d'étudier des problèmes régissant ce type d'écoulements, ceci a motivé notre intérêt pour le problème de Stokes compressible avec une loi d'état générale.

Ce manuscrit est organisé comme suit :

Dans le chapitre 2, on s'intéresse à l'étude de l'équation de transport (qui apparaît dans le système de Stokes). On montre l'existence d'une solution faible de ce problème en passant à la limite sur des schémas numériques. Ce résultat est démontré avec une condition faible sur le champ de vitesse. Un résultat analogue d'existence (sans utiliser un schéma numérique) avec les mêmes conditions de régularité sur la donnée du problème fait l'objet de l'arcticle fondateur de R. DiPerna et P.-L. Lions [21]. Ce papier donne aussi un résultat d'unicité.

Dans le chapitre 3, on traite le problème de Stokes stationnaire compressible avec une loi d'état de la forme $p = \varphi(\rho)$ (p :la pression, ρ :la densité et $\varphi \in C(\mathbb{R}, \mathbb{R})$, super-linéaire, convexe et croissante). On démontre l'existence de solution du problème en passant à la limite sur le schéma numérique quand le pas du maillage tend vers zéro. Le travail présenté dans ce chapitre généralise l'article [6], où un résultat similaire est démontré avec la loi d'état : $\varphi(\rho) = \rho^\gamma$,$\gamma > 1$ (voir aussi [13]). Le fait de considérer une loi d'état générale induit des difficultés supplémentaires, en particulier pour avoir les estimations sur la solution discrète et dans la passage à la limite sur l'EOS, on imite ici quelques idées développées dans [14], [8] ou [15] dans l'étude des équations de Navier-Stokes. Ce chapitre fait l'objet de l'arcticle [10] et avec des hypothèses différentes [9].

Dans le chapitre 4, contrairement aux chapitres précédents, on travaille sur le plan continu. On donne une preuve de l'existence de solution pour le problème de Stokes stationnaire compressible en utilisant une approximation par viscosité. Le résultat est démontré en passant à la limite sur un problème régularisé. Ceci consiste essentiellement en deux parties : l'étude du problème de Convection-Diffusion (qui apparaît dans le problème régularisé) où on démontre l'existence et l'unicité de solution et une deuxième partie où on présente le passage à la limite sur le problème régularisé.

Quelques notions préliminaires

Les équations de Navier-Stokes sont obtenues en appliquant les lois de conservation à un volume élémentaire Ω (un ouvert borné de $\mathbb{R}^d$, $d = 2, 3$) d'un fluide :

• **Conservation de la masse** :

$$\int_\Omega \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) = 0$$

où ρ représente la densité du fluide à l'instant t et à la position X et $\boldsymbol{u}$ représente la vitesse.

• **Conservation de la quantité de mouvement** :

$$\int_\Omega \frac{\partial \rho \boldsymbol{u}}{\partial t} + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \int_{\partial\Omega} \sigma \boldsymbol{n} = \int_\Omega \rho \boldsymbol{f}$$

où $\partial\Omega$ est le bord de Ω, $\boldsymbol{n}$ la normale sortante de Ω, σ est le tenseur des contraintes de l'écoulement et $\boldsymbol{f}$ est la densité massique des forces appliquées au fluide.

• **Conservation de l'énergie** :

$$\int_\Omega \frac{\partial \rho E}{\partial t} + \mathrm{div}(\rho E \boldsymbol{u}) - \int_{\partial\Omega} (\phi - \sigma \boldsymbol{u}) \cdot \boldsymbol{n} = \int_\Omega \rho \boldsymbol{f} \cdot \boldsymbol{u}$$

où ϕ est le flux de chaleur et E est l'énergie spécifique totale : $E = e + \frac{1}{2}\boldsymbol{u}^2$ où e est l'énergie spécifique interne.

En supposant que toutes ces fonctions de t et de X sont suffisamment régulières, on obtient en utilisant le théorème de divergence :

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) &= 0 \\ \frac{\partial \rho \boldsymbol{u}}{\partial t} + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}\sigma &= \rho \boldsymbol{f} \\ \frac{\partial \rho E}{\partial t} + \mathrm{div}(\rho E \boldsymbol{u}) + \mathrm{div}(\phi - \sigma \boldsymbol{u}) &= \rho \boldsymbol{f} \cdot \boldsymbol{u} \end{aligned}$$

Supposons maintenant que le fluide est newtonien. C'est à dire qu'il existe deux réels λ et μ (appelés coefficients de Lamé), tels que :

$$\begin{aligned} \sigma &= \tau - pI_d \\ \tau &= \lambda \mathrm{div}(\boldsymbol{u}) I_d + 2\mu D(\boldsymbol{u}) \end{aligned}$$

où

$$D(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)$$

et p est la pression, τ le tenseur des contraintes visqueuses, I_d la matrice identité et $D(\boldsymbol{u})$ est appelé le tenseur des déformations de l'écoulement. Enfin, en supposant que le fluide suit la loi de Fourier :

$$\phi = -k \nabla T$$

où k est la conductivité thermique et T la température, on obtient le système :

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) &= 0 \\ \frac{\partial \rho \boldsymbol{u}}{\partial t} + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \Delta \boldsymbol{u} + \nabla(p - (\lambda + \mu)\mathrm{div}\boldsymbol{u}) &= \rho \boldsymbol{f} \\ \frac{\partial \rho E}{\partial t} + \mathrm{div}(\rho E \boldsymbol{u}) - \mathrm{div}(k \nabla T) - \mathrm{div}(\tau \boldsymbol{u}) + \mathrm{div}(p \boldsymbol{u}) &= \rho \boldsymbol{f} \cdot \boldsymbol{u} \end{aligned}$$

que l'on doit compléter par une loi d'état pour relier toutes les grandeurs thermodynamiques.

Dans le cas d'un écoulement isovolume, isotherme et sans échange de chaleur dans un domaine borné régulier, l'équation de conservation de l'énergie devient inutile, et on a :

• $\nabla\rho = 0$ (écoulement homogène)

• $\mathrm{div}\boldsymbol{u} = 0$ (1) (écoulement isovolume)

(1) est aussi appelée condition d'incompressibilité.

En effet un écoulement est dit incompressible si le volume du fluide demeure constant sous l'action d'une pression externe, ceci se traduit mathématiquement par une masse volumique constante et donc l'équation de conservation de la masse prend la forme particulièrement simple (1).

Dans ce cas on obtient le système suivant :

$$\mathrm{div}\boldsymbol{u} = 0 \tag{1.0.1}$$

$$\rho(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) - \mu\Delta\boldsymbol{u} + \nabla p = \rho\boldsymbol{f} \tag{1.0.2}$$

Pour obtenir les équations de Navier-Stokes incompressibles, il nous reste à adimensionner les équations (1.0.1) et (1.0.2).On fixe alors une échelle de temps t_0, une échelle d'espace l_0 et une taille caractéristique f_0 pour les forces appliquées à l'écoulement. On en déduit une vitesse caractéristique u_0 :

$$u_0 = \frac{l_0}{t_0}$$

Et on pose :

$$t^* = \frac{t}{t_0}, \quad x^* = \frac{x}{l_0}, \quad \boldsymbol{u}^* = \frac{\boldsymbol{u}}{u_0}, \quad \boldsymbol{f}^* = \frac{\boldsymbol{f}}{f_0}, \quad p^* = \frac{p}{\rho u_0^2}$$

On obtient alors en omettant les "*"

$$\mathrm{div}\,\boldsymbol{u} = 0$$

$$\rho\left(\frac{u_0}{t_0}\frac{\partial \boldsymbol{u}}{\partial t} + \frac{u_0^2}{l_0}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) - \mu\frac{u_0}{l_0^2}\Delta\boldsymbol{u} + \frac{\rho u_0^2}{l_0}\nabla p = \rho f_0\boldsymbol{f}$$

i.e.

$$\mathrm{div}\,\boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{\mu}{\rho u_0 l_0}\Delta\boldsymbol{u} + \nabla p = \frac{f_0}{u_0/t_0}\boldsymbol{f}$$

Le système dépend des deux paramètres suivants :

• Le nombre de Reynolds :

$$\mathcal{R}e = \frac{\rho u_0 l_0}{\mu}$$

- Le nombre de Froude :

$$\mathcal{F}r = \frac{\rho u_0/t_0}{f_0}$$

On obtient alors :

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= 0 \\ \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + \nabla p &= \frac{1}{\mathcal{F}r}\boldsymbol{f} \end{aligned}$$

En prenant $\mathcal{F}r = 1$, on obtient les équations de Navier-Stokes :

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= 0 \\ \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \end{aligned}$$

Dans le cas d'un écoulement compressible (ρ non constante), les équations de Navier-Stokes peuvent se présenter (en négligeant les termes sources) comme suit :

$$\frac{\partial \rho}{dt} + \operatorname{div}(\rho \boldsymbol{u}) = 0 \tag{1.0.3a}$$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \operatorname{div}(\tau(\boldsymbol{u})) = 0 \tag{1.0.3b}$$

$$\frac{\partial \rho E}{\partial t} + \operatorname{div}(\rho E \boldsymbol{u}) + \operatorname{div}(p\boldsymbol{u}) + \operatorname{div}(-\kappa \nabla e) = \operatorname{div}(\tau(\boldsymbol{u}) \cdot \boldsymbol{u}) \tag{1.0.3c}$$

$$p = \varphi(\rho, e),\ E = \frac{1}{2}|\boldsymbol{u}|^2 + e \tag{1.0.3d}$$

Dans le cas barotrope (p ne dépend que de ρ), le système se réduit à :

$$\frac{\partial \rho}{dt} + \operatorname{div}(\rho \boldsymbol{u}) = 0 \tag{1.0.4a}$$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \operatorname{div}(\tau(\boldsymbol{u})) = 0 \tag{1.0.4b}$$

$$p = \varphi(\rho) \tag{1.0.4c}$$

Les équations de Stokes compressible s'obtiennent à partir des équations de Navier-Stokes en négligeant les termes non linéaires et peuvent se présenter comme suit :

$$\frac{\partial \boldsymbol{u}}{\partial t} - \mu \Delta \boldsymbol{u} - \frac{\mu}{3}\boldsymbol{\nabla}(\operatorname{div}\boldsymbol{u}) + \boldsymbol{\nabla} p = \boldsymbol{f}, \qquad (\mu > 0) \tag{1.0.5a}$$

$$\frac{\partial \rho}{dt} + \operatorname{div}(\rho \boldsymbol{u}) = 0 \tag{1.0.5b}$$

$$p = \varphi(\rho) \tag{1.0.5c}$$

Dans notre travail on s'intéresse au cas stationnaire, en tenant en compte les effets de la gravité (le terme source dépend de la densité) avec une loi d'état générale. Le problème de Stokes stationnaire compressible se présente comme suit :

$$-\mu\Delta\boldsymbol{u} - \frac{\mu}{3}\boldsymbol{\nabla}(\mathrm{div}\boldsymbol{u}) + \boldsymbol{\nabla}p = \nabla p = \boldsymbol{f} + \rho\boldsymbol{g}, \tag{1.0.6a}$$

$$\mathrm{div}(\rho\boldsymbol{u}) = 0, \tag{1.0.6b}$$

$$p = \varphi(\rho). \tag{1.0.6c}$$

Chapitre 2

Discrétisation de l'équation de transport avec un champ peu régulier

2.1 Motivation

Soit Ω un ouvert borné polyédrique de $\mathbb{R}^d$ (Ω polygone de $\mathbb{R}^2$, polyèdre de $\mathbb{R}^3$). On veut résoudre l'équation de transport linéaire suivante :

$$\left\{\begin{array}{l} \rho_t(x,t) + \operatorname{div}(\rho\boldsymbol{u})(x,t) = 0, (x,t) \in \Omega\times]0,T[\\ \rho(x,0) = \rho_0(x), x \in \Omega \end{array}\right. \tag{2.1.1}$$

avec $\rho_0 \in L^\infty(\Omega)$ et $\boldsymbol{u} \in L^1(0,T;W_0^{1,1}(\Omega))$ tq $(\operatorname{div}\boldsymbol{u})^- \in L^\infty(\Omega)$.

Définition 1 *Soit $Q = \Omega \times (0,T)$, on dit que $\rho \in L^\infty(Q)$ est solution faible du problème (2.1.1), si :*

$$\int_Q (\rho\psi_t + \boldsymbol{u}\rho\cdot\nabla\psi)\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t} + \int_\Omega \rho_0(x)\psi(x,0)\ \mathrm{d}\boldsymbol{x} = 0, \forall\psi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^+,\mathbb{R}) \tag{2.1.2}$$

On va montrer l'existence d'une solution faible du problème (2.1.1), en passant à la limite sur des schémas numériques. On considère alors un maillage polyédrique $\mathcal{T}$, on note par h_K le diamètre de K et par ε et ε_K l'ensemble des arrêtes intérieures de $\mathcal{T}$ et K respectivement (K mailles de$\mathcal{T}$) et pour la discrétisation en temps : $t_0 = 0 \le t_1 \le ... \le t_{N-1} = T$ et $k_n = t_n - t_{n-1}$.

Supposons que :$\boldsymbol{u}\in C_c^1(\Omega)$ et$\rho \in C^1(]0,T[\times\Omega)$ solution régulière de :

$$\rho_t(x,t) + \operatorname{div}(\rho\boldsymbol{u})(x,t) = 0.$$

En intégrant l'équation sur $K\in\mathcal{T}$ et en utilisant le schéma d'Euler implicite pour l'approximation de ρ_t, on obtient l'équation de bilan semi-discrétisée en temps suivante :

$$\frac{1}{k_n}\int_K (\rho(x,t_{n+1}) - \rho(x,t_n))\ \mathrm{d}\boldsymbol{x} + \int_K (\operatorname{div}\rho\boldsymbol{u})(x,t)\ \mathrm{d}\boldsymbol{x} = 0$$

On introduit les inconues discrètes $(\rho_K^n)_{K\in\mathcal{T},n\in\mathbb{N}}$, censées être des approximations de $\oint_K \rho(x,t_n)\,\mathrm{d}\boldsymbol{x}$, et le flux discret par interface σ, $F_{K,\sigma}^n$, sensé approcher le flux continu *i.e.*

$$\int_{t_n}^{t_{n+1}}\int_\sigma \rho u(x,t)\cdot n_{K\sigma} d\gamma(x).$$

L'équation discrète s'écrit alors :

$$\frac{\mid K\mid}{k_n}(\rho_K^{n+1}-\rho_K^n)+\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} F_{K,\sigma}^{n+1}=0$$

Pour définir le schéma numérique, il faut donc préciser l'expression de $F_{K,\sigma}^n$ en fonction des inconnues ρ_K^n . La vitesse u étant donnée, on pose :

$$v_{K\sigma}^{n(\pm)}=\frac{1}{k_n\mid\sigma\mid}\int\limits_{t_n}^{t_{n+1}}\int\limits_\sigma(\boldsymbol{u}\cdot n_{K\sigma})^{\pm}\ d\gamma(x)\,\mathrm{d}\boldsymbol{t}$$

où, pour $a\in\mathbb{R}$, a^+ (respectivement a^-) désigne la partie positive (respectivement négative) c'est à dire : $a^+=max(a,0)$ et $a^-=(-a)^+$.

On approche le flux continu en prenant le choix décentré amont de ρ sur σ par rapport à la vitesse u et implicite ou explicite en temps :

$$\text{schéma implicite : } F_{K,\sigma}^n=k_n\mid\sigma\mid(\rho_K^{n+1}v_{K\sigma}^{n(+)}-\rho_L^{n+1}v_{K\sigma}^{n(-)})$$

$$\text{schéma explicite : } F_{K,\sigma}^n=k_n\mid\sigma\mid(\rho_K^{n}v_{K\sigma}^{n(+)}-\rho_L^{n}v_{K\sigma}^{n(-)})$$

Ce choix est dit "upwind fort", et n'est pas celui qui est implanté en pratique.

En efffet, en pratique, il est plus facile d'imlpanter le choix suivant, dit "upwind faible"

$$\tilde{F}_{K,\sigma}^n=k_n\mid\sigma\mid(\rho_K^{n+1}(\tilde{v}_{K\sigma}^n)^+-\rho_L^{n+1}(\tilde{v}_{K\sigma}^n)^-)$$

avec :

$$\tilde{v}_{K\sigma}^{n}=\frac{1}{k_n\mid\sigma\mid}\int\limits_{t_n}^{t_{n+1}}\int\limits_\sigma(\boldsymbol{u}\cdot n_{K\sigma})\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}.$$

2.2 Schéma Upwind fort implicite

Le schéma est donné par :

$$\mid K\mid(\rho_K^{n+1}-\rho_K^n)+\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n\mid\sigma\mid\ (\rho_K^{n+1}v_{K\sigma}^{n(+)}-\rho_L^{n+1}v_{K\sigma}^{n(-)})=0,\ \forall K\in\mathcal{T} \tag{2.2.1}$$

2.2.1 Positivité de la solution

Nous allons montrer une propriété très importante, si la condition initiale ρ_0 est positive, alors la solution donnée par le schéma numérique reste positive.

Lemma 2.2.1 *Soit $(\rho_K^n)_{K\in\mathcal{T}}$ telle que : $\rho_K^n \geq 0$ pour tout $K \in \mathcal{T}$ et ρ_K^{n+1} solution de (2.2.1), alors on a*

$$\rho_K^{n+1} \geq 0, \forall K \in \mathcal{T}.$$

► Preuve du lemme 2.2.1 : Soit $M = card(\mathcal{T})$, on a :

$$\rho_K^{n+1} - \rho_K^n = \frac{-k_n}{\mid K \mid} \sum_{\sigma\in\varepsilon_K} \mid \sigma \mid (\rho_K^{n+1} v_{K\sigma}^{n(+)} - \rho_L^{n+1} v_{K\sigma}^{n(-)}), \forall K \in \mathcal{T}$$

$$\Rightarrow \rho_K^{n+1}(\mid K \mid + k_n) \sum_{\sigma\in\varepsilon_K} \mid \sigma \mid v_{K\sigma}^{n(+)} - \sum_{\sigma\in\varepsilon_K} \mid \sigma \mid k_n v_{K\sigma}^{n(-)} = \mid K \mid \rho_K^n, \forall K \in \mathcal{T}$$

On obtient alors un système linéaire de M équations $\grave{a}$ M inconnues $(\rho_K^{n+1})_{K\in\mathcal{T}}$.Les équations de ce système sont données par

$$\sum_{L\in\mathcal{T}} a_{K,L}\rho_L^{n+1} = b_K^n, \ \forall K \in \mathcal{T}, \tag{2.2.2}$$

avec

$$a_{K,K} = \mid K \mid + \sum_{\sigma\in\varepsilon_K} k_n \mid \sigma \mid v_{K\sigma}^{n(+)},$$

$$a_{K,L} = -k_n \mid \sigma \mid v_{K\sigma}^{n(-)}, \text{ avec } \sigma = \partial K \cap \partial L,$$

$$a_{K,L} = 0, \text{ si } \partial K \cap \partial L = \emptyset,$$
$$b_K = |K|\rho_K^n.$$

La matrice $A = (a_{K,L})_{K,L\in\mathcal{T}}$ saistfait les propriétés suivantes :

$$\begin{cases} a_{K,K} \geq 0 \\ a_{K,L} \leq 0 \\ a_{K,K} + \sum_{\sigma=K|L} a_{L,K} \geq 0 \end{cases}$$

En utilisant ces propriétès, on peut montrer que le système (2.2.2) admet une unique solution $(\rho_K^{n+1})_{K\in\mathcal{T}}$ et que $\rho_K^n \geq 0, \forall K \in \mathcal{T} \Rightarrow \rho_K^{n+1} \geq 0, \forall K \in \mathcal{T}$, ce résultat fait l'objet du lemme 3.8.4 dans le chapitre 3.

2.2.2 Estimation L^∞

Lemma 2.2.2 *Soit* $(\rho_K^{n+1})_{K\in\mathcal{T}}$ *la solution du schéma* (2.2.1), *supposons que :*

$$\forall n \in \{0...N-1\} : \int\limits_{t_n}^{t_{n+1}} \varphi(t)\ \mathrm{d}\boldsymbol{t} \leq \frac{1}{2} \tag{2.2.3}$$

avec

$$\varphi(t) = \|(\operatorname{div}\boldsymbol{u})^-\|_{L^\infty(\Omega)}$$

alors :

$$\forall n \in \{0...N-1\},\ M^n = \sup_{K\in\mathcal{T}} \rho_K^n \leq M^0 e^{2\int\limits_0^{t_n}\varphi(t)\ \mathrm{d}\boldsymbol{t}} \leq M^0 e^{2\int\limits_0^{T}\varphi(t)\ \mathrm{d}\boldsymbol{t}}.$$

► Preuve du lemme 2.2.2 : Montrons le par récurrence :

→ $n = 0$, c'est vrai vu que $t_0 = 0$

→ Hypothèse de récurrence : supposons que

$$M^n \leq M^0\, e^{2\int\limits_0^{t_n}\varphi(t)\ \mathrm{d}\boldsymbol{t}}$$

→ Montrons le pour n+1

$$\Rightarrow \rho_K^{n+1}(1 + \frac{k_n}{\mid K\mid}\sum_{\sigma\in\varepsilon_K} \mid\sigma\mid v_{K\sigma}^{n(+)}) - \sum_{\sigma\in\varepsilon_K}\mid\sigma\mid \frac{k_n}{\mid K\mid}\rho_K^{n+1}v_{K\sigma}^{n(-)} = \rho_K^n$$

$$\Rightarrow M^{n+1}(1 + \frac{k_n}{\mid K\mid}\sum_{\sigma\in\varepsilon_K}\mid\sigma\mid v_{K\sigma}^{n(+)} - v_{K\sigma}^{n(-)}) \leq M^n$$

$$\Rightarrow M^{n+1}(1 + \frac{1}{\mid K\mid}\int\limits_{t_n}^{t_{n+1}}\int\limits_K \operatorname{div}\boldsymbol{u}\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}) \leq M^n$$

$$\Rightarrow M^{n+1}(1 - \int\limits_{t_n}^{t_{n+1}} \|(\operatorname{div}\boldsymbol{u})^-\|_{L^\infty}\ \mathrm{d}\boldsymbol{t}) \leq M^n$$

Or, on peut facilement démontrer que $\forall\alpha\in\mathbb{R}_+$ telle que $0\leq\alpha\leq\frac{1}{2}$,

$$\frac{1}{1-\alpha} \leq 1 + 2\alpha$$

on obtient alors en utilisant l'hypothèse (2.2.3)

$$M^{n+1} \leq M^n(1 + 2\int_{t_n}^{t^n+1}\varphi(t)\ \mathrm{d}\boldsymbol{t})$$

$$\Rightarrow ln(M^{n+1}) \leq ln(M^n) + ln(1 + 2\int_{t_n}^{t_{n+1}}\varphi(t)\ \mathrm{d}\boldsymbol{t})$$

$$\Rightarrow ln(M^{n+1}) \leq ln(M^0) + 2\int_0^{t_{n+1}} \varphi(t)\ \mathrm{d}\boldsymbol{t}$$

$$\Rightarrow M^{n+1} \leq M^0\, e^{2\int_0^{t_{n+1}} \varphi(t)\ \mathrm{d}\boldsymbol{t}}.$$

2.2.3 Convergence du schéma

Dans cette étape on considère une suite de maillages $(\mathcal{T}_m, k_m)_{m\in\mathbb{N}}$ et on va passer à la limite quand m tend vers $+\infty$. Soit $h_m = \max_{K\in\mathcal{T}_m} h_K$, supposons que cette suite est régulière au sens de la définition suivante :

Définition 2 *Une suite $(\mathcal{T}_m, k_m)_{m\in\mathbb{N}}$ de maillages de Ω est dite régulière si :*

1. *$h_m \to 0$ as $m \to +\infty$,*
2. *$k_m \to 0$ as $m \to +\infty$,*
3. *there exists $\theta_0 > 0$ such that $\theta_m \geq \theta_0$, $\forall m \in \mathbb{N}$, with θ_m defined by*

$$\theta = \inf\ \{\frac{\xi_K}{h_K},\ K \in \mathcal{T}\} \tag{2.2.4}$$

avec ξ_K le diamètre de la plus grande boule incluse dans K.

On va démontrer le résultat suivant

Theorem 2.2.3 *Soit $(\mathcal{T}_m, k_m)_{m\in\mathbb{N}}$ une suite de maillages régulière au sens de la définition 2. On note par $\rho_{\mathcal{T}_m,k_m}$ la solution du schéma 2.2.1 qui correspond au maillage $(\mathcal{T}_m, k_m)$. Supposons que la vitesse $\boldsymbol{u}$ satisfait la condition* (2.2.3)*, on a alors :*

1. *$\rho_{\mathcal{T}_m k_m} \to \rho$ qd $m \to +\infty$, pour la topologie L^∞faible $\star$ (après extraction d'une sous suite),*
2. *ρ est solution du problème* (2.1.2)*.*

Preuve La convergence de la suite $(\rho_{\mathcal{T}_n,k_n})_{m\in\mathbb{N}}$ est une conséquence directe de l'estimation démontrée dans le lemme 2.2.2. On passe maintenant à la preuve de convergence du schéma. Soit $\Psi \in C_c^\infty(\Omega \times [0,T[)$, posons : $\Psi_K^n = \Psi(x_K, t_n), x_K \in K$, fixé.

Multiplions le schéma par Ψ_K^n et sommons sur toutes les mailles $K \in \mathcal{T}_m$ et pour $n \in \{0...N-1\}$, on obtient :

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m} |K|(\rho_K^{n+1} - \rho_K^n)\Psi_K^n + \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^{n+1}v_{K\sigma}^{n(+)} - \rho_L^{n+1}v_{K\sigma}^{n(-)})\Psi_K^n = 0.$$

On commence par le premier terme $T_1 = \displaystyle\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m} |K|(\rho_K^{n+1} - \rho_K^n)\Psi_K^n$, on va montrer que $T_1 \to -\int_0^T \int_\Omega \rho\Psi_t - \int_\Omega \rho_0(x)\Psi(x,0)\,\mathrm{d}\boldsymbol{x};\ h_m, k_m \to 0.$

En effet, on a :

$$T_1 = \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_m} |K| \rho_K^n \Psi_K^{n-1} - \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} |K| \rho_K^n \Psi_K^n$$

$$= \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_m} |K| \rho_K^n (\Psi_K^{n-1} - \Psi_K^n) - \sum_{K\in\mathcal{T}_m} |K| \rho_K^0 \Psi_K^0$$

$$= -\sum_{n=1}^{N} \sum_{K\in\mathcal{T}_m} |K| \rho_K^n \int_{t_{n-1}}^{t_n} \Psi_t(x_K, t)\,\mathrm{d}\boldsymbol{t} - \sum_{K\in\mathcal{T}_m} |K| \rho_K^0 \Psi_K^0$$

$$= -\sum_{n=1}^{N} \sum_{K\in\mathcal{T}_m} \int_K \int_{t_{n-1}}^{t_n} \rho_K^n \Psi_t(x,t)\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t} - \int_K \rho_K^0(x)\Psi(x,0)\,\mathrm{d}\boldsymbol{x} + R_1 + R_2$$

$$= -\int_0^T \int_\Omega \rho_{\mathcal{T}_m,k_m} \Psi_t\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t} - \int_\Omega \rho_K^0(x)\Psi(x,0)\,\mathrm{d}\boldsymbol{x} + R_1 + R_2$$

avec : $\rho_{\mathcal{T}_m,k_m} = \rho_K^{n+1}$ sur $K \times (t_n, t_{n+1})$

$$R_1 = \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_m} \int_K \int_{t_{n-1}}^{t_n} \rho_K^n (\Psi_t(x,t) - \Psi_t(x_K,t))\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}$$

$$\textit{et} : R_2 = \sum_{K\in\mathcal{T}_m} \int_K \rho_K^0 (\Psi(x,0) - \Psi(x_K,0))\,\mathrm{d}\boldsymbol{x}$$

On peut démontrer facilement que :

$$|R_1|, |R_2| \le C(\Omega, \Psi, T, \rho_0) h_m \to 0; h_m \to 0$$

On obtient alors, en utilisant le résultat 1 du théorème 2.2.3 que

$$T_1 \to -\int_0^T \int_\Omega \rho(x,s)\Psi_t(x,s)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s - \int_\Omega \rho_0(x)\Psi(x,0)\,\mathrm{d}\boldsymbol{x}. \tag{2.2.5}$$

On passe maintenant au deuxième terme T_2

$$T_2 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K, \sigma=K\backslash L} k_n |\sigma| (\rho_K^{n+1} v_{K\sigma}^{n(+)} - \rho_L^{n+1} v_{K\sigma}^{n(-)}) \Psi_K^n$$

on va montrer que

$$T_2 \to_{h_m,k_m\to 0} -\int_0^T \int_\Omega (\rho\boldsymbol{u})(x,t) \cdot (\bigtriangledown \Psi)(x,t)\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}$$

En effet,

$$T_2 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K, \sigma=K\backslash L} k_n |\sigma| (\rho_K^{n+1} v_{K\sigma}^{n(+)} - \rho_L^{n+1} v_{K\sigma}^{n(-)}) \Psi_K^n$$

$$= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\Psi_K^n - \Psi_{K\sigma}^{n(+)})\rho_K^{n+1} v_{K\sigma}^{n(+)}$$

$$- \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\Psi_K^n - \Psi_{K\sigma}^{n(-)})\rho_L^{n+1} v_{K\sigma}^{n(-)}$$

avec : $\Psi_{K\sigma}^{n(\pm)} = \dfrac{\int_{t_n}^{t_{n+1}} \int_\sigma \Psi(u\cdot n_{K\sigma})^{\pm}}{k_n|\sigma| v_{K\sigma}^{n(\pm)}}$, ce qui donne

$$T_2 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\Psi_K^n - \Psi_{K\sigma}^{n(+)})\rho_K^{n+1} v_{K\sigma}^{n(+)}$$
$$- \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\Psi_K^n - \Psi_{K\sigma}^{n(-)})\rho_K^{n+1} v_{K\sigma}^{n(-)} + R_3$$

avec

$$R_3 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\rho_K^{n+1} - \rho_L^{n+1})(\Psi_K^n - \Psi_{K\sigma}^{n(-)}) v_{K\sigma}^{n(-)}$$

on obtient alors,

$$T_2 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|\rho_K^{n+1}\Psi_K^n \tilde{v}_{K\sigma}^n$$
$$- \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|\rho_K^{n+1}(\Psi_{K\sigma}^{n(+)} v_{k\sigma}^{n(+)} - \Psi_{K\sigma}^{n(-)} v_{k\sigma}^{n(-)}) + R_3$$
$$= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \int_{t_n}^{t_{n+1}} \int_K \rho_K^{n+1}\Psi_K^n \operatorname{div}\boldsymbol{u} \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}$$
$$- \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n\rho_K^{n+1} \int_{t_n}^{t_{n+1}} \int_\sigma \Psi\boldsymbol{u}\cdot n_{K\sigma} \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t} + R_3$$

$$= \int_0^T \int_\Omega \rho_{\mathcal{T}_m,k_m}\Psi_{\mathcal{T}_m,k_m} \operatorname{div}\boldsymbol{u} \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t} - \int_0^T \int_\Omega \rho_{\mathcal{T}_m,k_m}\Psi_{\mathcal{T}_m,k_m} \operatorname{div}(\Psi u) \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t} + R_3$$

avec : $\Psi_{\mathcal{T}_m,k_m} = \Psi(x_K, t_n) \ sur K \times (t_n, t_{n+1})$,

supposons, connu, que $R_3 \to 0$ qd $m \to +\infty$, on aura alors :

$$T_2 \to \int_0^T \int_\Omega (\rho\Psi)(x,s) \operatorname{div}\boldsymbol{u}(x,t) \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t} - \int_0^T \int_\Omega (\rho)(x,t) \operatorname{div}(\Psi\boldsymbol{u})(x,t) \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}$$

i.e on a démontré que

$$T_2 \to -\int_0^T \int_\Omega (\rho u)(x,s)\cdot(\bigtriangledown\Psi)(x,t) \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t} \qquad (2.2.6)$$

On a finalement, les résultats (2.2.5) et (2.2.6) donnent que la limte ρ est solution du problème (2.1.2). Reste à montrer que $R_3 \to 0$ qd $m \to +\infty$:

$$|R_3| = |\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^{n+1}-\rho_L^{n+1})(\Psi_K^n - \Psi_{K\sigma}^{n(-)})v_{K\sigma}^{n(-)}|$$

$$\leq C(\Psi)\, h \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\,|\rho_K^{n+1}-\rho_L^{n+1}|\,(v_{K\sigma}^{n(+)}+v_{K\sigma}^{n(-)})$$

Posons $\bar{v}_{k\sigma}^n = v_{K\sigma}^{n(+)}+v_{K\sigma}^{n(-)}$, on obtient alors par l'inégalité de Cauchy-Schwarz :

$$|R_3| \leq C(\Psi)\, h \underbrace{(\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\bar{v}_{k\sigma}^n)^{\frac{1}{2}}}_{R_{3,1}}$$

$$\underbrace{(\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\bar{v}_{k\sigma}^n(\rho_K^{n+1}-\rho_L^{n+1})^2)^{\frac{1}{2}}}_{R_{3,2}}$$

Pour avoir une estimation sur le terme $R_{3,2}$, on revient au schéma, on multiplie par ρ_K^{n+1} et on somme sur toutes les mailles $K \in \mathcal{T}$ et pour $n \in \{0...N-1\}$, on obtient alors :

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|(\rho_K^{n+1}-\rho_K^n)\rho_K^{n+1} + \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^{n+1}v_{K\sigma}^{n(+)} - \rho_L^{n+1}v_{K\sigma}^{n(-)})\rho_K^{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n+1}-\rho_K^n)^2}{2} + \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n+1})^2}{2} - \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n})^2}{2}$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^{n+1}v_{K\sigma}^{n(+)} - \rho_L^{n+1}\ v_{K\sigma}^{n(-)})\rho_K^{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n+1})^2}{2} - \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n})^2}{2}$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^{n+1}v_{K\sigma}^{n(+)} - \rho_L^{n+1}\ v_{K\sigma}^{n(-)})\rho_K^{n+1} \leq 0$$

$$\Rightarrow \sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{N})^2}{2} - \sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{0})^2}{2}$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\rho_K^{n+1}v_{K\sigma}^{n(+)}(\frac{\rho_K^{n+1}+\rho_L^{n+1}}{2}+\frac{\rho_K^{n+1}-\rho_L^{n+1}}{2})$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\rho_K^{n+1}v_{K\sigma}^{n(-)}(\frac{\rho_K^{n+1}+\rho_L^{n+1}}{2}-\frac{\rho_K^{n+1}-\rho_L^{n+1}}{2})\leq 0$$

$$\Rightarrow \sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^N)^2}{2}-\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^0)^2}{2}+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\rho_K^{n+1}\tilde{v}_{K\sigma}^n(\frac{\rho_K^{n+1}+\rho_L^{n+1}}{2})$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\rho_K^{n+1}(\frac{\rho_K^{n+1}-\rho_L^{n+1}}{2})\bar{v}_{k\sigma}^n\leq 0$$

$$\Rightarrow \sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^N)^2}{2}-\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^0)^2}{2}+\sum_{n=0}^{N-1}\sum_{\sigma\in\varepsilon,\sigma=K\backslash L}\frac{k_n|\sigma|}{2}((\rho_K^{n+1})^2-(\rho_L^{n+1})^2)\tilde{v}_{K\sigma}^n$$

$$+\sum_{n=0}^{N-1}\sum_{\sigma\in\varepsilon,\sigma=K\backslash L}\frac{k_n|\sigma|}{2}(\rho_K^{n+1}-\rho_L^{n+1})^2\bar{v}_{k\sigma}^n\leq 0$$

$$\Rightarrow \sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^N)^2}{2}-\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^0)^2}{2}+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L}\frac{(\rho_K^{n+1})^2}{2}\int_{t_n}^{t_{n+1}}\int_\sigma u\cdot n_{K\sigma}$$

$$+\sum_{n=0}^{N-1}\sum_{\sigma\in\varepsilon,\sigma=K\backslash L}\frac{k_n|\sigma|}{2}(\rho_K^{n+1}-\rho_L^{n+1})^2\bar{v}_{k\sigma}^n\leq 0$$

$$\Rightarrow \sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^N)^2}{2}-\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^0)^2}{2}+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\frac{(\rho_K^{n+1})^2}{2}\int_{t_n}^{t_{n+1}}\int_K \mathrm{div}\boldsymbol{u}\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}$$

$$+\sum_{n=0}^{N-1}\sum_{\sigma\in\varepsilon,\sigma=K\backslash L}\frac{k_n|\sigma|}{2}(\rho_K^{n+1}-\rho_L^{n+1})^2\bar{v}_{k\sigma}^n\leq 0$$

$$\Rightarrow \sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^N)^2}{2}-\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^0)^2}{2}+\int_0^T\int_\Omega\frac{(\rho_{\mathcal{T}k})^2}{2}\mathrm{div}\boldsymbol{u}\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}$$

$$+\sum_{n=0}^{N-1}\sum_{\sigma\in\varepsilon,\sigma=K\backslash L}\frac{k_n|\sigma|}{2}(\rho_K^{n+1}-\rho_L^{n+1})^2\bar{v}_{k\sigma}^n\leq 0$$

Ce qui donne

$$\sum_{n=0}^{N-1}\sum_{\sigma\in\varepsilon,\sigma=K\backslash L}\frac{k_n|\sigma|}{2}(\rho_K^{n+1}-\rho_L^{n+1})^2\bar{v}_{k\sigma}^n\leq C(\rho_0,u) \tag{2.2.7}$$

Pour le deuxième terme $R_{3,1}$, on a :

$$R_{3,1} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K, \sigma=K\setminus L} k_n|\sigma|\bar{v}_{k\sigma}^n = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K, \sigma=K\setminus L} \int_{t_n}^{t_{n+1}} \int_\sigma |\boldsymbol{u}\cdot n_{K\sigma}|$$

$$= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}\|_{L^1(\partial K)}$$

en adaptant le Lemme 2.3 dans [11] au cas $W^{1,1}$, on a $\exists C$ dépendant de θ_0 t.q :

$$R_{3,1} \leq \frac{C}{h} \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t_n}^{t_{n+1}} \|\boldsymbol{u}\|_{W^{1,1}(K)} = \frac{C}{h}\|\boldsymbol{u}\|_{L^1(0,T;W^{1,1}(\Omega))} \tag{2.2.8}$$

Et par conséquent, les deux inégalités (2.2.7) et (2.2.8) donnent

$$|R_3| \leq C(\Psi, \rho_0, \boldsymbol{u})h^{\frac{1}{2}} \to 0; h, k \to 0$$

ceci termine la preuve.

2.3 Schéma Upwind fort explicite

Le schéma est donnée par :

$$| K | (\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma\in\varepsilon_K, \sigma=K\setminus L} k_n | \sigma | \ (\rho_K^n v_{K\sigma}^{n(+)} - \rho_L^n v_{K\sigma}^{n(-)}) = 0, \forall K \in \mathcal{T} \tag{2.3.1}$$

2.3.1 Positivité de la solution sous condition CFL

Lemma 2.3.1 *Soit $(\rho_K^n)_{K\in\mathcal{T}}$ telle que : $\rho_K^n \geq 0$ pour tout $K \in \mathcal{T}$ et ρ_K^{n+1} solution de* (2.3.1), *alors on a*

$$\rho_K^{n+1} \geq 0, \ \forall K \in \mathcal{T},$$

sous la condition CFL suivante :

$$\forall n \in \{0...N-1\}, k_n \leq \frac{| K |}{\sum_{\sigma\in\varepsilon_K} | \sigma | \ v_{K\sigma}^{n(+)}} \tag{2.3.2}$$

▶ Preuve du lemme 2.3.1 : On a :

$$\rho_K^{n+1} = \rho_K^n - \frac{1}{| K |} \sum_{\sigma\in\varepsilon_K, \sigma=K\setminus L} k_n | \sigma | \ (\rho_K^n v_{K\sigma}^{n(+)} - \rho_L^n v_{K\sigma}^{n(-)})$$

$$\Rightarrow \rho_K^{n+1} = \rho_K^n(1 - \frac{1}{| K |} \sum_{\sigma\in\varepsilon_K, \sigma=K\setminus L} k_n | \sigma | \ v_{K\sigma}^{n(+)})$$

$$+\frac{1}{\mid K\mid}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n\mid\sigma\mid\rho_L^n v_{K\sigma}^{n(-)}$$

la condition CFL (2.3.2) donne :

$$(1-\frac{1}{\mid K\mid}\sum_{\sigma\in\varepsilon_K} k_n\mid\sigma\mid v_{K\sigma}^{n(+)})\geq 0$$

on obtient finalement,

$$\rho_K^{n+1},\ \forall K\in\mathcal{T}.$$

2.3.2 Estimation L^∞

Lemma 2.3.2 *Soit* $(\rho_K^{n+1})_{K\in\mathcal{T}}$ *la solution du schéma* (2.3.1)*, alors :*

$$\forall n\in\{0...N-1\}, M^n=\sup_{K\in\mathcal{T}}\rho_K^n\leq C(\Omega,u)\|\rho_0\|_{L^\infty}$$

► Preuve du lemme 2.3.2 :

On a :

$$\rho_K^{n+1}=\rho_K^n(1-\frac{1}{\mid K\mid}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n\mid\sigma\mid v_{K\sigma}^{n(+)})+\frac{1}{\mid K\mid}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L,\sigma=K\setminus L} k_n\mid\sigma\mid\rho_L^n v_{K\sigma}^{n(-)}$$

$$\Rightarrow M^{n+1}\leq M^n(1-\frac{1}{\mid K\mid}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n\mid\sigma\mid v_{K\sigma}^{n(+)})$$

$$+\frac{M^n}{\mid K\mid}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L,\sigma=K\setminus L} k_n\mid\sigma\mid\ v_{K\sigma}^{n(-)}$$

$$\Rightarrow M^{n+1}\leq M^n(1-\frac{1}{\mid K\mid}\int_{t_n}^{t_{n+1}}\int_K \mathrm{div}u\ \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t})$$

$$\Rightarrow M^{n+1}\leq M^n(1+\frac{1}{\mid K\mid}\int_{t_n}^{t_{n+1}}\int_K (\mathrm{div}u)^-\ \ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t})$$

$$\leq M^n(1+\|(\mathrm{div}u)^-\|_{L^1(L^\infty)})$$

On obtient alors :

$$\forall n\in\{0,N-1\}, M^n\leq C(\Omega,u)\|\rho_0\|_\infty.$$

2.3.3 Convergence du schéma

Theorem 2.3.3 *Soit $(\mathcal{T}_m, k_m)_{m\in\mathbb{N}}$ une suite de maillages régulière au sens de la définition 2. On note par $\rho_{\mathcal{T}_m,k_m}$ la solution du schéma* (2.3.1) *qui correspond au maillage $(\mathcal{T}_m, k_m)$.*

Supposons que $\forall n \in \{0...N-1\}$, k_n satisfait la condition CFL (2.3.2) *et la condition suivante :*

$$k_n \leq \frac{|K|}{2\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} \mid \sigma \mid v_{K\sigma}^{n(-)}} \tag{2.3.3}$$

on a alors :

1. *$\rho_{\mathcal{T}_m k_m} \to \rho$ qd $m \to +\infty$, pour la topologie L^∞faible $\star$ (après extraction d'une sous suite),*
2. *ρ est solution du problème* (2.1.2).

Preuve

La convergence de la suite $(\rho_{\mathcal{T}_m k_m})_{m\in\mathbb{N}}$ résulte du lemme 2.3.2. On va maintenant démontrer la convergence du schema.

On multiplie le schéma par Ψ_K^n et on somme sur toutes les mailles $K \in \mathcal{T}_m$ et pour $n = 0..N-1$, on obtient alors :

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}|K|(\rho_K^{n+1}-\rho_K^n)\Psi_K^n + \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^n v_{K\sigma}^{n(+)} - \rho_L^n v_{K\sigma}^{n(-)})\Psi_K^n = 0$$

On a déjà montré dans la preuve du thòrème 2.3.3 que :

$$T_1 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}|K|(\rho_K^{n+1}-\rho_K^n)\Psi_K^n \to -\int_0^T\int_\Omega \rho\Psi_t - \int_\Omega \rho_0(x)\Psi(x,0)\,\mathrm{d}\boldsymbol{x};\ h,k\to 0$$

Reste à montrer que :

$$T_2 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^n v_{K\sigma}^{n(+)} - \rho_L^n v_{K\sigma}^{n(-)})\Psi_K^n$$

$$\to -\int_0^T\int_\Omega (\rho u)(x,t)\cdot(\bigtriangledown\Psi)(x,t)\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t};\ h,k\to 0$$

On a :

$$T_2 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\Psi_K^n - \Psi_{K\sigma}^{n(+)})\rho_K^n v_{K\sigma}^{n(+)}$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\Psi_K^n - \Psi_{K\sigma}^{n(-)})\rho_L^n v_{K\sigma}^{n(-)}$$

$$= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\rho_K^n \Psi_K^n v_{k\sigma}^n$$

$$-\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\rho_K^n (\Psi_{K\sigma}^{n(+)} v_{k\sigma}^{n(+)} - \Psi_{K\sigma}^{n(-)} v_{k\sigma}^{n(-)}) + R_1$$

$$avec : R_1 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^n - \rho_L^n)(\Psi_K^n - \Psi_{K\sigma}^{n(-)}) v_{K\sigma}^{n(-)}$$

$$T_2 = \int_0^T \rho_{\mathcal{T}_m,k_m} \Psi_{\mathcal{T}_m,k_m} \mathrm{div} u \ \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{t} - \int_0^T \rho_{\mathcal{T}_m,k_m} \Psi_{\mathcal{T}_m,k_m} \mathrm{div}(\Psi u) \ \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{t} + R_1$$

et

$$\rho_{\mathcal{T}_m k_m} = \rho_K^n \ sur K \times (t_n, t_{n+1})$$

On aura alors :

$$T_2 \to \int_0^T \int_\Omega (\rho\Psi)(x,t) \mathrm{div} u(x,t) \ \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{t} - \int_0^T \int_\Omega (\rho)(x,t) \mathrm{div}(\Psi u)(x,t) \ \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{t}$$

$$= -\int_0^T \int_\Omega (\rho u)(x,t) \cdot (\bigtriangledown \Psi)(x,t) \ \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{t}$$

Reste à montrer que

$$R_1 \to 0; h, k \to 0$$

en effet, on a :

$$|R_1| = |\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_m} \sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^n - \rho_L^n)(\Psi_K^n - \Psi_{K\sigma}^{n(-)}) v_{K\sigma}^{n(-)}|$$

$$\leq C(\Psi)\, h \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|\, |\rho_K^n - \rho_L^n|\, \bar{v}_{K\sigma}^n$$

par l'inégalité de Cauchy-Schwarz, on a :

$$|R_1| \leq C(\Psi)\, h \underbrace{(\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma| \bar{v}_{k\sigma}^n)^{\frac{1}{2}}}_{R_{1,1}}$$

$$\underbrace{(\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma| \bar{v}_{k\sigma}^n (\rho_K^n - \rho_L^n)^2)^{\frac{1}{2}}}_{R_{1,2}}$$

Pour le premier terme $R_{1,1}$, comme dans la preuve du théorème2.2.3, on a $\exists C$ indépendante du maillage t.q :

$$R_{1,1} \leq \frac{C}{h}\|\boldsymbol{u}\|_{L^1(0,T;W^{1,1}(\Omega))} \tag{2.3.4}$$

Pour avoir une estimation sur le terme $R_{1,2}$, on revient au schéma et on multiplie par ρ_K^n, on obtient :

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|(\rho_K^n-\rho_K^n)\rho_K^{n+1}+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L}k_n|\sigma|(\rho_K^n v_{K\sigma}^{n(+)}-\rho_L^n v_{K\sigma}^{n(-)})\rho_K^n=0$$

$$\Rightarrow -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n+1}-\rho_K^n)^2}{2}+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n+1})^2}{2}-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^n)^2}{2}$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L}k_n|\sigma|\rho_K^n v_{K\sigma}^{n(+)}(\frac{\rho_K^n+\rho_L^n}{2}+\frac{\rho_K^n-\rho_L^n}{2})$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L}k_n|\sigma|\rho_K^n v_{K\sigma}^{n(-)}(\frac{\rho_K^n+\rho_L^n}{2}-\frac{\rho_K^n-\rho_L^n}{2})=0$$

$$\Rightarrow -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n+1}-\rho_K^n)^2}{2}+\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^N)^2}{2}-\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^0)^2}{2}+\int_0^T\int_\Omega\frac{(\rho_{\mathcal{T}k})^2}{2}\mathrm{div}u$$

$$+\sum_{n=0}^{N-1}\sum_{\sigma\in\varepsilon,\sigma=K\setminus L}\frac{k_n|\sigma|}{2}(\rho_K^{n+1}-\rho_L^{n+1})^2\bar{v}_{K\sigma}^n=0$$

et donc,

$$\sum_{n=0}^{N-1}\sum_{\sigma\in\varepsilon,\sigma=K\setminus L}\frac{k_n|\sigma|}{2}(\rho_K^{n+1}-\rho_L^{n+1})^2\bar{v}_{K\sigma}^n\leq C(\Omega,\rho_0,u)+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|\frac{(\rho_K^{n+1}-\rho_K^n)^2}{2} \tag{2.3.5}$$

En utilisant le schéma (2.3.1) le second terme dans (2.3.5) donne :

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|(\rho_K^{n+1}-\rho_K^n)^2=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|[\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L}\frac{k_n\mid\sigma\mid}{|K|}(\rho_K^n v_{K\sigma}^{n(+)}-\rho_L^n v_{K\sigma}^{n(-)})]^2$$

$$=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\frac{1}{|K|}[\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L}k_n\mid\sigma\mid(\rho_K^n v_{k\sigma}^n+(\rho_K^n-\rho_L^n)v_{k\sigma}^{n(-)}]^2$$

$$\leq 2\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\frac{1}{|K|}[\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L}k_n\mid\sigma\mid\rho_K^n v_{k\sigma}^n]^2$$

$$+2\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\frac{1}{|K|}[\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L}k_n\mid\sigma\mid(\rho_K^n-\rho_L^n)v_{k\sigma}^{n(-)}]^2$$

$$\leq 2\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\frac{1}{|K|}[\int_{t_n}^{t_{n+1}}\int_K \rho_K^n \mathrm{div}u\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}]^2$$

$$+2\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\frac{1}{|K|}[\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n \mid \sigma \mid (\rho_K^n-\rho_L^n)v_{k\sigma}^{n(-)}]^2$$

$$\leq C(\Omega,\rho_0,u)$$

$$+2\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}[\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n \mid \sigma \mid (\rho_K^n-\rho_L^n)^2 v_{k\sigma}^{n(-)}][\frac{1}{|K|}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n \mid \sigma \mid v_{K\sigma}^{n(-)}]$$

à ce stade on utilise la condition(2.3.3), on obtient alors

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|(\rho_K^{n+1}-\rho_K^n)^2 \leq C(\Omega,\rho_0,u)+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n \mid \sigma \mid (\rho_K^n-\rho_L^n)^2 v_{k\sigma}^{n(-)}$$

i.e

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}|K|(\rho_K^{n+1}-\rho_K^n)^2 \leq C(\Omega,\rho_0,u)+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n \mid \sigma \mid (\rho_K^n-\rho_L^n)^2 \bar{v}_{K\sigma}^{n} \quad (2.3.6)$$

on a finalement, (2.3.5) et (2.3.6) donnent :

$$|R_{1,2}| \leq C(\Psi,\rho_0,\boldsymbol{u}) \quad (2.3.7)$$

et donc (2.3.7) et (2.3.4)

$$|R_1| \leq C(\Psi,\rho_0,\boldsymbol{u})h^{\frac{1}{2}} \to 0; h,k\to 0,$$

ceci termine la preuve.

2.4 Schéma Upwind faible implicite

Le schéma est donné par :

$$\mid K \mid (\rho_K^{n+1}-\rho_K^n)+\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n \mid \sigma \mid \ (\rho_K^{n+1}(\tilde{v}_{K\sigma}^n)^+ - \rho_L^{n+1}(\tilde{v}_{K\sigma}^n)^-)=0,\ \forall K\in\mathcal{T} \quad (2.4.1)$$

avec :

$$\tilde{v}_{K\sigma}^n=\frac{1}{k_n \mid \sigma \mid}(\int_{t_n}^{t_{n+1}}\int_\sigma (u\cdot n_{K\sigma})\ d\gamma(x)\,\mathrm{d}\boldsymbol{t})$$

2.4.1 Positivité de la solution

En supposant que : $\forall K\in\mathcal{T},\forall n\in\{0,N-1\},\rho_K^n\geq 0$, on a alors la solution ρ_K^{n+1} du schéma (2.4.1) est positive ($\rho_K^{n+1}\geq 0$) et la preuve est identique à celle du lemme 2.2.1.

2.4.2 Estimation L^{∞}

On rappelle le lemme qui donne l'estimation :

Lemma 2.4.1 *Soit* $(\rho_K^{n+1})_{K\in\mathcal{T}}$ *solution du schéma* (2.4.1)*, supposons que :*

$$\forall n \in \{0...N-1\} : \int_{t_n}^{t_{n+1}} \varphi(t) \;\mathrm{d}\boldsymbol{t} \leq \frac{1}{2} \tag{2.4.2}$$

avec

$$\varphi(t) = \|(\mathrm{div}\boldsymbol{u})^{-}\|_{L^{\infty}(\Omega)}$$

alors :

$$\forall n \in \{0...N-1\},\; M^n = \sup_{K\in\mathcal{T}} \rho_K^n \leq M^0 e^{2\int_0^{t_n} \varphi(t)\;\mathrm{d}\boldsymbol{t}} \leq M^0 e^{2\int_0^{T} \varphi(t)\;\mathrm{d}\boldsymbol{t}}$$

Preuve La preuve est la même donnée dans le lemme 2.2.2.

2.4.3 Convergence du schéma

Theorem 2.4.2 *Soit* $(\mathcal{T}_m, k_m)_{m\in\mathbb{N}}$ *une suite de maillages régulière au sens de la définition 2. On note par* $\rho_{\mathcal{T}_m,k_m}$ *la solution du schéma* (2.4.1) *qui correspond au maillage* $(\mathcal{T}_m, k_m)$*.*
Supposons que la vitesse $\mathbf{u}$ *satisfait la condition* (2.4.2)*, on a alors :*

1. $\rho_{\mathcal{T}_m k_m} \to \rho$ *qd* $m \to +\infty$ *, pour la topologie* L^{∞}*faible* $\star$
   (après extraction d'une sous suite),
2. ρ *est solution du problème* (2.1.2)*.*

Preuve
On multiplie le schéma par Ψ_K^net on somme sur toutes les mailles $K \in \mathcal{T}_m$ et pour $n \in \{0...N-1\}$, on obtient :

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m} |K|(\rho_K^{n+1} - \rho_K^n)\Psi_K^n + \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma=K\backslash L} k_n|\sigma|(\rho_K^{n+1}(\tilde{v}_{K\sigma}^n)^{+} - \rho_L^{n+1}(\tilde{v}_{K\sigma}^n)^{-})\Psi_K^n = 0$$

On a, d'après les paragraphes précédents

$$T_1 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m} |K|(\rho_K^{n+1} - \rho_K^n)\Psi_K^n \to -\int_0^T\int_\Omega \rho\Psi_t - \int_\Omega \rho_0(x)\Psi(x,0)\,\mathrm{d}\boldsymbol{x};\; h_m, k_m \to 0$$

Montrons maintenant que :

$$T_2 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n|\sigma|(\rho_K^{n+1}(\tilde{v}_{K\sigma}^n)^{+} - \rho_L^{n+1}(\tilde{v}_{K\sigma}^n)^{-})\Psi_K^n$$

$$\to -\int_0^T \int_\Omega (\rho u)(x,t)\cdot(\bigtriangledown\Psi)(x,t)\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t};\ \ h_m,k_m\to 0$$

On a :

$$T_2=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\Psi_K^n-\Psi_{K\sigma}^n)\rho_K^{n+1}(\tilde{v}_{K\sigma}^n)^+$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\Psi_K^n-\Psi_{K\sigma}^n)\rho_L^{n+1}(\tilde{v}_{K\sigma}^n)^-$$

$$=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|\Psi_K^n\rho_K^{n+1}\tilde{v}_{K\sigma}^n-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K} k_n|\sigma|\Psi_{K\sigma}^n\rho_K^{n+1}\tilde{v}_{K\sigma}^n+R_1$$

avec :

$$R_1=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\rho_K^{n+1}-\rho_L^{n+1})(\Psi_K^n-\Psi_{K\sigma}^n)(\tilde{v}_{K\sigma}^n)^-$$

et $R_1\to 0; h_m,k_m\to 0$, la preuve est identique à celle du théorème 2.2.3, et donc

$$T_2=\int_0^T \rho_{\mathcal{T}k}\Psi_{\mathcal{T}_m,k_m}\mathrm{div}u\ \ \mathrm{d}\boldsymbol{x}\ \ \mathrm{d}\boldsymbol{t}-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K}\Psi_{K\sigma}^n\rho_K^{n+1}\int_{t_n}^{t_{n+1}}\int_\sigma u\cdot n_{K\sigma}\,d\gamma(x)dt+R_1$$

$$T_2=\int_0^T \rho_{\mathcal{T}k}\Psi_{\mathcal{T}_m,k_m}\mathrm{div}u\ \ \mathrm{d}\boldsymbol{x}\ \ \mathrm{d}\boldsymbol{t}-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K}\Psi_{K\sigma}^n\rho_K^{n+1}\int_{t_n}^{t_{n+1}}\int_\sigma w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}+R_1$$

avec : $w\in\mathbb{P}^1$, approximation Crouzeix-Raviart de $\boldsymbol{u}$.

On définit $\Psi_{K\sigma}^n$ par :

$$\Psi_{K\sigma}^n=\begin{cases}\Psi(x_\sigma,t_n),\ \text{si } w\cdot n_{K\sigma}\text{ change de signe}\\ \dfrac{\int_{t_n}^{t_{n+1}}\int_\sigma \Psi w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}}{\int_{t_n}^{t_{n+1}}\int_\sigma w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}},\ \text{si } w\cdot n_{K\sigma}\text{ ne change pas de signe}\end{cases}$$

$$T_2=\int_0^T \rho_{\mathcal{T}_m,k_m}\Psi_{\mathcal{T}k}\mathrm{div}u\ \ \mathrm{d}\boldsymbol{x}\ \ \mathrm{d}\boldsymbol{t}-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K\cap E}\int_{t_n}^{t_{n+1}}\int_\sigma \Psi(x_\sigma,s_n)\rho_K^{n+1}w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K\setminus E}\int_{t_n}^{t_{n+1}}\int_\sigma \Psi\rho_K^{n+1}w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}+R_1$$

avec :$E=\{\sigma\in\varepsilon$ telle que : $w\cdot n_{K\sigma}$ change de signe$\}$

$$T_2=\int_0^T \rho_{\mathcal{T}k}\Psi_{\mathcal{T}_m,k_m}\mathrm{div}u\ \ \mathrm{d}\boldsymbol{x}\ \ \mathrm{d}\boldsymbol{t}-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K\cap E}\int_{t_n}^{t_{n+1}}\int_\sigma \Psi\rho_K^{n+1}w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K\setminus E}\int_{t_n}^{t_{n+1}}\int_\sigma \Psi\rho_K^{n+1}w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}+R_1+R_2$$

avec :

$$R_2=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K\cap E}\int_{t_n}^{t_{n+1}}\int_\sigma(\Psi-\Psi(x_\sigma,s_n))\rho_K^{n+1}w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}$$

alors :

$$T_2\to\int_0^T\int_\Omega(\rho\Psi)(x,t)\ \mathrm{div}\ u(x,t)\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}-\int_0^T\int_\Omega\rho(x,t)\mathrm{div}(\Psi u)(x,t)\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}.$$

Reste à montrer que

$$R_2\to 0,\ h_m,k_m\to 0 \tag{2.4.3}$$

on aura besoin du lemme suivant :

Lemma 2.4.3 *Soit w une fonction affine sur K telle que : $\{w=0\}\cap\partial K\neq\emptyset$, alors on a :*

$$\|w\|_{L^1(\partial K)}\le 3\frac{h^2}{s}\|\nabla w\|_{L^1(K)}$$

avec : h et s diamètre et surface de K respectivement.

►Preuve du lemme 2.4.3 :

On peut supposer que : $w\neq 0$ telle que : $\|w\|_\infty=1$.

On a : $\exists a\in K$ et $b\in\partial K$ tels que : $|w(a)|=1$ et $w(b)=0$

on a alors :

$$|\nabla w|\ge|\nabla w\cdot n|\ge\frac{1}{d(a,b)}$$

donc :

$$\|\nabla w\|_{L^1(K)}\ge\frac{s}{h}$$

d'autre part, on a :

$$\|\nabla w\|_{L^1(\partial K)}=\sum_{\sigma_i\in\partial K}\int_{\sigma_i}|w|\le 3h$$

d'où le résultat :

$$\|\nabla w\|_{L^1(\partial K)}\le\frac{3h^2}{s}\|\nabla w\|_{L^1(K)}.$$

On démontre maintenant (2.4.3), on a :

$$R_2=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap E}\int_{t_n}^{t_{n+1}}\int_\sigma(\Psi-\Psi(x_\sigma,s_n))\rho_K^{n+1}w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}\to 0;h,k\to 0$$

On a :

$$|R_2|=|\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap E}\int_{t_n}^{t_{n+1}}\int_\sigma(\Psi-\Psi(x_\sigma,s_n))\rho_K^{n+1}w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}|$$

$$
\begin{aligned}
&\leq C(h+k)\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K\cap E}\int_{t_n}^{t_{n+1}}\int_\sigma |w|\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}\\
&\leq C(h+k)\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\int_{t_n}^{t_{n+1}}\|w\|_{L^1(K)}\\
&\leq C(h+k)\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\int_{t_n}^{t_{n+1}}\|w\|_{W^{1,1}(K)}\\
&\leq C(h+k)\|w\|_{L^1(0,T;W^{1,1}(\Omega))}\\
&\leq C(h+k)\|u\|_{L^1(0,T;W^{1,1}(\Omega))}\to 0; k,h\to 0
\end{aligned}
$$

(Stabilité du projecteur Crouzeix-Raviart)

Reste à montrer que :

$$\nabla_{\mathcal{T}} w \to \nabla u, \text{ dans } L^1(\Omega)$$

avec :$\nabla_{\mathcal{T}} w = \nabla w, \forall K\in\mathcal{T}$

Lemma 2.4.4 *Soit f une fonction de $L^1(\Omega)$ et $f_{\mathcal{T}}$ définie par :*

$$f_{\mathcal{T}} = \frac{1}{|K|}\int_K f(x)\,\mathrm{d}\boldsymbol{x}$$

On a alors : $\|f-f_{\mathcal{T}}\|_{L^1(\Omega)}\to 0, size(\mathcal{T})\to 0$

►Preuve du lemme 2.4.4 :

⋆Etape1 : Si $f\in C_c^1(\Omega)$

On a :

$$
\begin{aligned}
\forall x\in K : |f(x)-f_{\mathcal{T}}(x)| &= |f(x)-f(y)|, \text{ pour } y\in K \text{ (théorème de la moyenne)}\\
&\leq C|x-y|\leq C size(\mathcal{T})
\end{aligned}
$$

Donc :

$$\|f-f_{\mathcal{T}}\|_{L^1(\Omega)}\leq C|\Omega| size(\mathcal{T})\to 0, size(\mathcal{T})\to 0$$

⋆Etape2 : Si $f\in L^1(\Omega)$

On a : $\forall\varepsilon\geq 0\exists\Phi\in C_c^\infty(\Omega)$telle que : $\|f-\Phi\|_{L^1(\Omega)}\leq\varepsilon$

On aura alors : si size($\mathcal{T}$)$\leq\delta(\varepsilon)$:

$$
\begin{aligned}
\|f-f_{\mathcal{T}}\|_{L^1(\Omega)} &\leq \|f-\Phi\|_{L^1(\Omega)}+\|\Phi-\Phi_{\mathcal{T}}\|_{L^1(\Omega)}+\|\Phi_{\mathcal{T}}-f_{\mathcal{T}}\|_{L^1(\Omega)}\\
&\leq 2\|f-\Phi\|_{L^1(\Omega)}+\|\Phi-\Phi_{\mathcal{T}}\|_{L^1(\Omega)}\leq 3\varepsilon
\end{aligned}
$$

2.5 Schéma Upwind faible explicite

Le schéma est donné par :

$$|\,K\,|\,(\rho_K^{n+1}-\rho_K^n)+\sum_{\sigma\in\varepsilon_K,\sigma=K\backslash L} k_n\,|\,\sigma\,|\,\left(\rho_K^n(\tilde{v}_{K\sigma}^n)^+-\rho_L^n(\tilde{v}_{K\sigma}^n)^-\right)=0,\ \forall K\in\mathcal{T} \qquad (2.5.1)$$

2.5.1 Positivité de la solution

En supposant que : $\forall K \in \mathcal{T}, \forall n \in \{0, N-1\}, \rho_K^n \geq 0$, alors la solution ρ_K^{n+1} du schéma(2.5.1) est positive ($\rho_K^{n+1} \geq 0$), sous la condition CFL suivante :

$$\forall n \in \{0...N-1\}, k_n \leq \frac{\mid K \mid}{\sum_{\sigma \in \varepsilon_K} \mid \sigma \mid (\tilde{v}_{K\sigma}^n)^+} \tag{2.5.2}$$

La preuve est dientique à celle du lemme 2.3.1.

2.5.2 Estimation L^∞

De même que dans le paragraphe(2.3.1), on a si la condition CFL (2.5.2) est satisfaite :

$$\forall n \in \{0...N-1\}, M^n = \sup_{K \in \mathcal{T}} \rho_K^n \leq C(\Omega, \boldsymbol{u}) \|\rho_0\|_\infty.$$

2.5.3 Convergence du schéma

Theorem 2.5.1 *Soit $(\mathcal{T}_m, k_m)_{m \in \mathbb{N}}$ une suite de maillages régulière au sens de la définition 2. On note par $\rho_{\mathcal{T}_m, k_m}$ la solution du schéma* (2.5.1) *qui correspond au maillage $(\mathcal{T}_m, k_m)$.*

Supposons que $\forall n \in \{0...N-1\}$, k_n satisfait la condition CFL (2.5.2) *et la condition suivante :*

$$k_n \leq \frac{|K|}{2 \sum_{\sigma \in \varepsilon_K, \sigma = K \backslash L} \mid \sigma \mid (v_{K\sigma}^n)^-} \tag{2.5.3}$$

on a alors :

1. *$\rho_{\mathcal{T}_m k_m} \to \rho$ qd $m \to +\infty$, pour la topologie L^∞faible $\star$ (après extraction d'une sous suite),*
2. *ρ est solution du problème* (2.1.2)*.*

Preuve On multiplie le schéma par Ψ_K^n et on somme sur toutes les mailles $K \in \mathcal{T}_m$ et pour $n = 0..N-1$, on obtient :

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| (\rho_K^{n+1} - \rho_K^n) \Psi_K^n + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \varepsilon_K, \sigma = K \backslash L} k_n |\sigma| (\rho_K^n (\tilde{v}_{K\sigma}^n)^+ - \rho_L^n (\tilde{v}_{K\sigma}^n)^-) \Psi_K^n = 0$$

On a d'après les paragraphes précédents :

$$T_1 = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_m} |K| (\rho_K^{n+1} - \rho_K^n) \Psi_K^n \to -\int_0^T \int_\Omega \rho \Psi_t - \int_\Omega \rho_0(x) \Psi(x,0) \, \mathrm{d}\boldsymbol{x}; \;\; h_m, k_m \to 0$$

Pour montrer que :

$$T_2 = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_m} \sum_{\sigma \in \varepsilon_K, \sigma = K \backslash L} k_n |\sigma| (\rho_K^n (\tilde{v}_{K\sigma}^n)^+ - \rho_L^n (\tilde{v}_{K\sigma}^n)^-) \Psi_K^n$$

$$\to -\int_0^T \int_\Omega (\rho u)(x,s)\cdot(\bigtriangledown\Psi)(x,s)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s;\;\; h_m, k_m \to 0$$

En procédant de la même façon que dans les parties précédentes, on obtient :

$$T_2 = \int_0^T \int_\Omega \rho_{\mathcal{T}k}\Psi_{\mathcal{T}k}\mathrm{div}u \;\;\mathrm{d}\boldsymbol{x}\;\;\mathrm{d}\boldsymbol{t} - \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K\cap E}\int_{t_n}^{t_{n+1}}\int_\sigma \Psi\rho_K^n w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K\setminus E}\int_{t_n}^{t_{n+1}}\int_\sigma \Psi\rho_K^n w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t} + R_1 + R_2$$

avec : $R_1 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K,\sigma=K\setminus L} k_n|\sigma|(\rho_K^n-\rho_L^n)(\Psi_K^n-\Psi_{K\sigma}^n)(\tilde{v}_{K\sigma}^n)^-$

$$et : R_2 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_m}\sum_{\sigma\in\varepsilon_K\cap E}\int_{t_n}^{t_{n+1}}\int_\sigma (\Psi-\Psi(x_\sigma,s_n))\rho_K^n w\cdot n_{K\sigma}\,d\gamma(x)\,\mathrm{d}\boldsymbol{t}$$

tels que :

$$R_1 \to 0, \text{d'après le théorème}\, 2.3.3.$$

et :

$$R_2 \to 0, \text{d'après le théorème}\, 2.4.2.$$

ce qui donne finalement la convergence du schéma vers une solution du problème (2.1.2).

Ce travail a été étendu par F.Boyer avec $\boldsymbol{u} \in W^{1,1}(\Omega)$ et des conditions aux limites entrantes sur ρ, voir [12].

Chapitre 3

Discrétisation des équations de Stokes sationnaires compressibles

3.1 Abstract

In this chapter, we propose a discretization for the compressible Stokes problem with an equation of state of the form $p = \varphi(\rho)$ (where p stands for the pressure, ρ for the density and φ is a superlinear nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$). This scheme is based on Crouzeix-Raviart approximation spaces. The discretization of the momentum balance is obtained by the usual finite element technique. The discrete mass balance is obtained by a finite volume scheme, with an upwinding of the density, and two additional terms. We prove the existence of a discrete solution and the convergence of this approximate solution to a solution of the continuous problem.

3.2 Introduction

Let Ω be a bounded open set of $\mathbb{R}^d$, polygonal if $d = 2$ and polyhedral if $d = 3$. Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ be a convex nondecreasing function such that :

$$\varphi(0) = 0,\ \varphi \text{ is } C^1 \text{ on } \mathbb{R}_+^\star$$

and

$$\forall a \in \mathbb{R},\ \exists b > 0 \text{ such that : } \varphi(s) \geq as - b,\ \forall s \in \mathbb{R}_+. \tag{3.2.1}$$

For M, $\mu > 0$, $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$ and $\boldsymbol{g} \in L^\infty(\Omega)^d$, we consider the following problem :

$$-\mu\Delta\boldsymbol{u} - \frac{\mu}{3}\boldsymbol{\nabla}(\mathrm{div}\boldsymbol{u}) + \boldsymbol{\nabla}p = \boldsymbol{f} + \rho\boldsymbol{g} \text{ in } \Omega,\ \ \boldsymbol{u} = 0 \text{ on } \partial\Omega, \tag{3.2.2a}$$

$$\mathrm{div}(\rho\boldsymbol{u}) = 0 \ \text{ in } \Omega,\ \rho \geq 0 \ \text{ in } \Omega,\ \int_\Omega \rho(x)\,\mathrm{d}\boldsymbol{x} = M, \tag{3.2.2b}$$

$$p = \varphi(\rho) \text{ in } \Omega. \tag{3.2.2c}$$

Remark 3.2.1

- *We assume that the function φ is convex, but not necessarily strictly convex. We also assume that φ is nondecreasing but it can be constant on an interval (in fact, since φ is convex, the function φ is, at least for m large enough, increasing on $[m, +\infty)$).*
- *The condition* (3.2.1) *is equivalent to the following one :*

$$\liminf_{s \to +\infty} \varphi(s)/s = +\infty$$

- *The fact that $\varphi(0) = 0$ is not a restriction since p can be replaced by $(p - \varphi(0))$ in the momemtum equation and the EOS (namely the equation* (3.2.2c)*) can be written as $p - \varphi(0) = \varphi(\rho) - \varphi(0)$.*
- *The convexity of the function φ can be replaced by the following condition : there exist $a, \tilde{a}, b, \tilde{b} > 0$ and $\gamma > 1$ such that :*

$$\forall s \in \mathbb{R}_+,\ as^\gamma - b \leq \varphi(s) \leq \tilde{a}s^{2\gamma-1} + \tilde{b}. \tag{3.2.3}$$

  Here also the function φ is assumed to be nondecreasing but not necessarily increasing. We give more details using this condition in section 3.5.
- *The coefficient $\mu/3$ in the second term of the Left Hand Side of* (3.2.2a) *is natural from the physical point of view. From the mathematical point of view, it is easy to replace it by $\bar{\mu}$, as long as $\bar{\mu} \geq 0$.*

Definition 3.2.2 *Let $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$, $\boldsymbol{g} \in L^\infty(\Omega)^d$ and $M > 0$. A weak solution of Problem* (3.2.2) *is a function $(\boldsymbol{u}, p, \rho) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$ satisfying :*

$$\begin{aligned} \mu \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3} \int_\Omega \mathrm{div}(\boldsymbol{u})\mathrm{div}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \ \mathrm{div}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\ = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \textit{ for all } \boldsymbol{v} \in \mathrm{H}^1_0(\Omega)^d, \end{aligned} \tag{3.2.4a}$$

$$\int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi \,\mathrm{d}\boldsymbol{x} = 0 \textit{ for all } \varphi \in \mathrm{W}^{1,\infty}(\Omega), \tag{3.2.4b}$$

$$\rho \geq 0 \textit{ a.e. in } \Omega,\ \int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M,\ p = \varphi(\rho) \textit{ a.e. in } \Omega. \tag{3.2.4c}$$

The main objective of this paper is to present a numerical scheme for the computation of an approximate solution of Problem (3.2.2) and to prove the convergence (up to a subsequence, since, up to now, no uniqueness result is available for the solution of (3.2.2)) of this approximate solution towards a weak solution of (3.2.2) (*i.e.* a solution of (3.2.4)) as the mesh size goes to 0. The present paper follows a previous paper [6] where a similar result was presented in the case $\varphi(\rho) = \rho^\gamma$, $\gamma > 1$ (see also [13]). We present here a discretization with the so called Crouziex-Raviart element, as in [6]. However, it could be possible also, without additional difficulties, to use a MAC scheme, as in [7]. The fact to consider a general EOS (instead of $p = \rho^\gamma$) induces some additional difficulties with respect to the previous papers [6] and [7]. In particular for the estimates on the discrete solutions (Section 3.4.2 and

Appendix 3.6) and for passing to the limit in the EOS (Section 3.4.3 and Appendix 3.7). For passing in the limit in the EOS, we mimic some ideas which were developped for the study of the Navier-Stokes equations, see [14], [8] or [15]. A part of the results given in this paper was presented in the FVCA6 workshop (Prague, 2011) and in a short paper (containing few proofs) in the proceedings of this workshop, see [9]. The present paper is more general. In particular, it considers more general EOS and it includes the gravity effects (two improvements which induce the need of non trivial developments, for instance for obtaining estimates on u an p and for passing to the limit in the EOS). Furthermore, the present paper contains complete proofs and an appendix with lemmas interesting for their own sake.

Remark 3.2.3 *In the spirit of [14], [8] or [15] (which are devoted to the study of the compressible Navier-Stokes equations, but not on the discretization point of view), it is worth noticing that if* $(\rho, u) \in L^2(\Omega) \times \mathrm{H}^1_0(\Omega)$ *satisfies* (3.2.4b)*, then, it is known that* (ρ, u) *is a renormalized solution of* $\mathrm{div}(\rho u) = 0$ *in the sense of [4], that is*

$$(\rho\phi'(\rho) - \phi(\rho))\mathrm{div}(u) + \mathrm{div}(\phi(\rho)u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d),$$

for any C^1*-function* ϕ *from* $\mathbb{R}$ *to* $\mathbb{R}$ *such that* ϕ' *is bounded (in order to give a sense to the preceding equation, we set* $u = 0$ *in* $\mathbb{R}^d \setminus \Omega$*, so that* $u \in H^1(\mathbb{R}^d)$*). This is explained in Remark 3.7.3.*

3.3 Discrete spaces and numerical scheme

Let $\mathcal{T}$ be a decomposition of the domain Ω in simplices, which we call herafter a triangulation of Ω, regardless of the space dimension. By $\mathcal{E}(K)$, we denote the set of the edges ($d = 2$) or faces ($d = 3$) σ of the element $K \in \mathcal{T}$; for short, each edge or face will be called an edge hereafter. The set of all edges of the mesh is denoted by $\mathcal{E}$; the set of edges included in the boundary of Ω is denoted by $\mathcal{E}_{\mathrm{ext}}$ and the set of internal edges (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\mathrm{ext}}$) is denoted by $\mathcal{E}_{\mathrm{int}}$. The decomposition $\mathcal{T}$ is assumed to be regular in the usual sense of the finite element literature (*e.g.* [2]), and, in particular, $\mathcal{T}$ satisfies the following properties : $\bar{\Omega} = \bigcup_{K \in \mathcal{T}} \bar{K}$; if $K, L \in \mathcal{T}$, then $\bar{K} \cap \bar{L} = \emptyset$, $\bar{K} \cap \bar{L}$ is a vertex or $\bar{K} \cap \bar{L}$ is a common edge of K and L, which is denoted by $K|L$. For each internal edge of the mesh $\sigma = K|L$, $\boldsymbol{n}_{KL}$ stands for the normal vector of σ, oriented from K to L (so that $\boldsymbol{n}_{KL} = -\boldsymbol{n}_{LK}$). By $|K|$ and $|\sigma|$ we denote the (d and $d-1$ dimensional) measure, respectively, of an element K and of an edge σ, and h_K and h_σ stand for the diameter of K and σ, respectively. We measure the regularity of the mesh through the parameter θ defined by :

$$\theta = \inf \{\frac{\xi_K}{h_K},\ K \in \mathcal{T}\} \tag{3.3.1}$$

where ξ_K stands for the diameter of the largest ball included in K. Note that for all $\sigma \in \mathcal{E}_{\mathrm{int}}$, $\sigma = K|L$, we have $h_\sigma \geq \xi_K \geq \theta\, h_K$ and $h_\sigma \leq h_L$ and so $\theta\, h_K \leq h_L \leq \theta^{-1} h_K$. Note also that for all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}(K)$, the inequality $h_\sigma\, |\sigma| \leq 2\ \theta^{-d}\, |K|$ holds ([11, relation (2.2)]) and if $\sigma = K|L$ a rough estimate gives $|K| \leq (2/\theta)^{2d} |L|$. These relations will be used throughout this paper. Finally, as usual, we denote by h the quantity $\max_{K \in \mathcal{T}} h_K$.

The space discretization relies on the Crouzeix-Raviart element (see [3] for the seminal paper and, for instance, [5, pp. 199–201] for a synthetic presentation). The reference element is the unit d-simplex and the discrete functional space is the space P_1 of affine polynomials. The degrees of freedom are determined by the following set of edge functionals :

$$\{F_\sigma,\ \sigma \in \mathcal{E}(K)\}, \qquad F_\sigma(v) = |\sigma|^{-1} \int_\sigma v \,\mathrm{d}\gamma. \tag{3.3.2}$$

The mapping from the reference element to the actual one is the standard affine mapping. Finally, the continuity of the average value of a discrete functions v across each edge of the mesh, $F_\sigma(v)$, is required, thus the discrete space V_h is defined as follows :

$$\begin{aligned} V_h = \{&v \in L^2(\Omega)\ :\ \forall K \in \mathcal{T},\ v|_K \in P_1(K)\,; \\ &\forall \sigma \in \mathcal{E}_{\mathrm{int}},\ \sigma = K|L,\ \ F_\sigma(v|_K) = F_\sigma(v|_L);\ \forall \sigma \in \mathcal{E}_{\mathrm{ext}},\ F_\sigma(v) = 0\}. \end{aligned} \tag{3.3.3}$$

Indeed, this space V_h should be denoted by $V_{\mathcal{T}}$ since it depends on $\mathcal{T}$ and not only on h (which is given by $\mathcal{T}$) but this (somewhat incorrect) notation is commonly used.

The space of approximation for the velocity is the space $\boldsymbol{W}_h$ of vector-valued functions each component of which belongs to V_h : $\boldsymbol{W}_h = (V_h)^d$. The pressure and the density are approximated by the space L_h of piecewise constant functions :

$$L_h = \left\{q \in L^2(\Omega)\ :\ q|_K = \text{ constant, } \forall K \in \mathcal{T}\right\}.$$

Since only the continuity of the integral over each edge of the mesh is imposed, the functions of V_h are discontinuous through each edge ; the discretization is thus nonconforming in $H^1(\Omega)^d$. We then define, for $1 \leq i \leq d$ and $\boldsymbol{u} \in V_h$, $\partial_{h,i}\, \boldsymbol{u}$ as the function of $\mathrm{L}^2(\Omega)$ which is equal to the derivative of u with respect to the i^{th} space variable almost everywhere. This notation allows to define the discrete gradient, denoted by $\boldsymbol{\nabla}_h$, for both scalar and vector-valued discrete functions and the discrete divergence of vector-valued discrete functions, denoted by div_h.

The Crouzeix-Raviart pair of approximation spaces for the velocity and the pressure is *inf-sup* stable, in the usual sense for "piecewise H^1" discrete velocities, *i.e.* there exists $c_{\mathrm{i}} > 0$ only depending on Ω and, in a nonincreasing way, on θ, such that :

$$\forall p \in L_h, \qquad \sup_{\boldsymbol{v} \in \boldsymbol{W}_h} \frac{\displaystyle\int_\Omega p\ \mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{v}\|_{1,b}} \geq c_{\mathrm{i}}\, \|p - m(p)\|_{\mathrm{L}^2(\Omega)},$$

where $m(p)$ is the value of p over Ω and $\|\cdot\|_{1,b}$ stands for the broken Sobolev H^1 semi-norm, which is defined for scalar as well as for vector-valued functions by :

$$\|v\|_{1,b}^2 = \sum_{K \in \mathcal{T}} \int_K |\boldsymbol{\nabla} v|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega |\boldsymbol{\nabla}_h v|^2 \,\mathrm{d}\boldsymbol{x}.$$

This norm is known to control the L^2 norm by a Poincaré inequality (*e.g.* [5, lemma 3.31]). We also define a discrete semi-norm on L_h, similar to the usual H^1 semi-norm used in the finite volume context :

$$\forall \rho \in L_h, \qquad |\rho|_{\mathcal{T}}^2 = \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}}, \\ \sigma = K|L}} \frac{|\sigma|}{h_\sigma}\, (\rho_K - \rho_L)^2.$$

From the definition (3.3.2), each velocity degree of freedom may be indexed by the number of the component and the associated edge, thus the set of velocity degrees of freedom reads :

$$\{v_{\sigma,i},\ \sigma \in \mathcal{E}_{\rm int},\ 1 \le i \le d\}.$$

We denote by e_σ the usual Crouzeix-Raviart shape function associated to σ, *i.e.* the scalar function of V_h such that $F_\sigma(e_\sigma) = 1$ and $F_{\sigma'}(e_\sigma) = 0$, for all $\sigma' \in \mathcal{E} \setminus \{\sigma\}$.

Similarly, each degree of freedom for the pressure is associated to a cell K, and the set of pressure degrees of freedom is denoted by $\{p_K,\ K \in \mathcal{T}\}$.

We define by r_h the following interpolation operator :

$$r_h : \quad \left| \begin{array}{lll} \mathrm{H}^1_0(\Omega) & \longrightarrow & V_h \\ u & \mapsto & r_h u = \displaystyle\sum_{\sigma\in\mathcal{E}} F_\sigma(u)\, e_\sigma = \sum_{\sigma\in\mathcal{E}} |\sigma|^{-1} \left(\int_\sigma v \,\mathrm{d}\gamma\right) e_\sigma. \end{array} \right. \tag{3.3.4}$$

This operator naturally extends to vector-valued functions (*i.e.* to perform the interpolation from $\mathrm{H}^1_0(\Omega)^d$ to $\boldsymbol{W}_h$) and we keep the same notation r_h for both the scalar and vector case. The properties of r_h are gathered in the following lemma. They are proven in [3].

Theorem 3.3.1 *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \ge \theta_0$, where θ is defined by* (3.3.1). *The interpolation operator r_h enjoys the following properties :*

1. *preservation of the divergence :*

$$\forall \boldsymbol{v} \in \mathrm{H}^1_0(\Omega)^d,\ \forall q \in L_h, \qquad \int_\Omega q\ {\rm div}_h(r_h \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega q\ {\rm div}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x},$$

2. *stability :*

$$\forall v \in \mathrm{H}^1_0(\Omega), \qquad \|r_h v\|_{1,b} \le c_1(\theta_0)\, |v|_{\mathrm{H}^1(\Omega)}\,,$$

3. *approximation properties :*

$$\begin{array}{l} \forall v \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega),\ \forall K \in \mathcal{T}, \\ \qquad \|v - r_h v\|_{\mathrm{L}^2(K)} + h_K\, \|\boldsymbol{\nabla}_h (v - r_h v)\|_{\mathrm{L}^2(K)} \le c_2(\theta_0)\, h_K^2\, |v|_{\mathrm{H}^2(K)}\,. \end{array}$$

In both above inequalities, the notation $c_i(\theta_0)$ means that the real number c_i only depends on θ_0 and Ω, and, in particular, does not depend on the parameter h characterizing the size of the cells ; this notation will be kept throughout the paper.

The following compactness result was proven in [11, Theorem 3.3].

Theorem 3.3.2 *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions satisfying the following assumptions :*

1. *$\forall n \in \mathbb{N}$, there exists a triangulation of the domain $\mathcal{T}_n$ such that $v_n \in V_{h_n}$, where V_{h_n} is the space of Crouzeix-Raviart discrete functions associated to $\mathcal{T}_n$ (and h_n given by $\mathcal{T}_n$), as defined by* (3.3.3)*, and the parameter θ_n characterizing the regularity of $\mathcal{T}_n$ is bounded away from zero independently of n,*

2. *the sequence* $(v_n)_{n\in\mathbb{N}}$ *is uniformly bounded with respect to the broken Sobolev* H^1 *semi-norm,* i.e. *:*

$$\forall n \in \mathbb{N}, \qquad \|v_n\|_{1,b} \le C,$$

where C *is a constant real number and* $\|\cdot\|_{1,b}$ *stands for the broken Sobolev* H^1 *semi-norm associated to* $\mathcal{T}_n$ *(with a slight abuse of notation, namely dropping, for short, the index* n *pointing the dependence of the norm with respect to the mesh).*

Then, when $n \to \infty$*, possibly up to the extraction of a subsequence, the sequence* $(v_n)_{n\in\mathbb{N}}$ *converges (strongly) in* $\mathrm{L}^2(\Omega)$ *to a limit* $\bar{v}$ *such that* $\bar{v} \in \mathrm{H}^1_0(\Omega)$*.*

We now present the numerical scheme we use. Let ρ^* be the mean density, *i.e.* $\rho^* = M/|\Omega|$ where $|\Omega|$ stands for the measure of the domain Ω. Let also α and ξ be given, with $\alpha > 0$ and $0 < \xi < 2$. Let $\mathcal{T}$ be a (regular) decomposition of the domain Ω in simplices. The discrete unknowns are $\boldsymbol{u}$, p and ρ, with $\boldsymbol{u} \in \boldsymbol{W}_h$ and $p, \rho \in L_h$. Using the notations previously introduced, we consider the following numerical scheme for the discretization of Problem (3.2.2) :

$$\begin{aligned}
\mu \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3}\int_\Omega \mathrm{div}_h(\boldsymbol{u})\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \;\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\
= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \;\text{ for all } \boldsymbol{v} \in \boldsymbol{W}_h,
\end{aligned} \tag{3.3.5a}$$

$$\sum_{\sigma=K|L} \big(|\sigma| \boldsymbol{u}^+_{K,\sigma}\, \rho_K - |\sigma| \boldsymbol{u}^-_{K,\sigma}\, \rho_L\big) + M_K + T_K = 0 \;\text{ for all } K \in \mathcal{T}, \tag{3.3.5b}$$

$$p_K = \varphi(\rho_K) \;\text{ for all } K \in \mathcal{T}. \tag{3.3.5c}$$

The quantity $\boldsymbol{u}_{K,\sigma}$ is defined by

$$\boldsymbol{u}_{K,\sigma} = |\sigma|^{-1} \int_\sigma \boldsymbol{u} \,\mathrm{d}\gamma \cdot \boldsymbol{n}_{KL}.$$

As usual, $\boldsymbol{u}^+_{K,\sigma} = \max(\boldsymbol{u}_{K,\sigma},\ 0)$ and $\boldsymbol{u}^-_{K,\sigma} = -\min(\boldsymbol{u}_{K,\sigma},\ 0)$, so that $\boldsymbol{u}_{K,\sigma} = \boldsymbol{u}^+_{K,\sigma} - \boldsymbol{u}^-_{K,\sigma}$. The terms M_K and T_K read :

$$M_K = h^\alpha\, |K|\, \left(\rho_K - \rho^*\right), \tag{3.3.6a}$$

$$T_K = \sum_{\sigma=K|L} h^\xi\, \frac{|\sigma|}{h_\sigma}\, \left(|\rho_K| + |\rho_L|\right) \left(\rho_K - \rho_L\right). \tag{3.3.6b}$$

3.4 Existence and convergence of approximate solutions

3.4.1 Existence of a solution

Let $\mathcal{T}$ be a (regular) decomposition of the domain Ω in simplices. We prove in this section the existence of a discrete solution, that the existence of a solution to (3.3.5), by using the Brouwer fixed point theorem to a convenient application T from $\mathbb{R}^N$ to $\mathbb{R}^N$ where $N =\mathrm{card}(\mathcal{T})$. We first define T.

Let $\tilde\rho = (\tilde\rho_K)_{K\in\mathcal{T}}$. Choosing the elements of $\mathcal{T}$ in an arbitrary order, we then have $\tilde\rho \in \mathbb{R}^N$. We calculate p by the following relation : $p_K = \varphi(\tilde\rho_K^+)$ for all $K \in \mathcal{T}$.

We now compute $\boldsymbol{u}$ as the unique solution (in $\boldsymbol{W}_h$) of (3.3.5a) with $\tilde\rho$ instead of ρ in the Right Hand Side of (3.3.5a) (and p given by $p_K = \varphi(\tilde\rho_K^+)$ for all $K \in \mathcal{T}$). The existence and uniqueness of $\boldsymbol{u}$ is an easy consequence of the coercivity in $\boldsymbol{W}_h$ of the bilinear form

$$(u, v) \mapsto \mu \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.$$

Furthermore, the solution $\boldsymbol{u}$ continuously depends on $\tilde\rho$ (since φ is continuous).

We have now to define ρ (and we will set $T(\tilde\rho) = \rho$). We change a little bit the term T_K. Instead of (3.3.6b), we take

$$T_K = \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\tilde\rho_K| + |\tilde\rho_L|) \, (\rho_K - \rho_L) \, .$$

With this choice of T_K, the set of Equations (3.3.5b) leads to the linear system of N equations with N unknowns (which are ρ_K for $K \in \mathcal{T}$). The equations of this system may be written as :

$$\sum_{L\in\mathcal{T}} a_{K,L}\rho_L = b_K \text{ for all } K \in \mathcal{T}, \tag{3.4.1}$$

with

$$a_{K,K} = h^\alpha |K| + \sum_{\sigma=K|L} \big(|\sigma| u^+_{K,\sigma} + h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\tilde\rho_K| + |\tilde\rho_L|)\,\big),$$

$$a_{K,L} = -|\sigma| u^-_{K,\sigma} - h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\tilde\rho_K| + |\tilde\rho_L|) \text{ if } \sigma = K|L,$$

$$a_{K,L} = 0 \text{ if } K \text{ and } L \text{ do not share an interface.}$$

$$b_K = h^\alpha |K| \rho^\star.$$

Using the fact that $u^-_{L,\sigma} = u^+_{K,\sigma}$ (for $\sigma = K|L$), one has, for all $K \in \mathcal{T}$,

$$\sum_{L\in\mathcal{T}} a_{K,L} > 0$$

and, for all $K, L \in \mathcal{T}$, $K \neq L$,

$$a_{K,L} \leq 0.$$

With these properties, it is quite easy to show thet the system (3.4.1) has a unique solution. Furthermore, since $b_K > 0$ for all $K \in \mathcal{T}$ the solution ρ satisfy $\rho_K > 0$ for all $K \in \mathcal{T}$ (see Lemma 3.8.4). Finally, since the coeffcients $a_{K,L}$ and b_K depend continuously of $\tilde\rho$ (and since the application $A \mapsto A^{-1}$ is continuous on the set of invertible $N \times N$ matrix), the solution ρ of (3.4.1) continuously depends on $\tilde\rho$.

We define now (as we said before) the map T from $\mathbb{R}^N$ to $\mathbb{R}^N$ setting $T(\tilde{\rho}) = \rho$. The map T is continuous.

If $\rho \in \mathrm{Im}(T)$, we also showed that $\rho_K > 0$ for all $K \in \mathcal{T}$. Futhermore summing for $K \in \mathcal{T}$ the equations (3.4.1) we obtain

$$\sum_{K\in\mathcal{T}} h^\alpha |K] \rho_K = \sum_{K\in\mathcal{T}} b_K = \sum_{K\in\mathcal{T}} h^\alpha |K| \rho^\star.$$

With the definition of $\rho^\star$, this gives $\sum_{K\in\mathcal{T}} |K|\rho_K = M$. Since $\rho \mapsto \sum_{K\in\mathcal{T}} |K||\rho_K|$ is a norm on $\mathbb{R}^N$, this proves that the whole set $\mathrm{Im}(T)$ is included in a fixed ball of $\mathbb{R}^N$. Then, we can apply the Brouwer fixed point theorem. It gives the existence of $\rho \in \mathbb{R}^N$ such that $T(\rho) = \rho$. This gives the existence of a solution $(\boldsymbol{u}, p, \rho)$ to (3.3.5).

We conclude this section by remarking that if $(\boldsymbol{u}, p, \rho)$ is a solution to (3.3.5), we necessarily have $T(\rho) = \rho$ and this show that

$$\rho_K > 0 \text{ for all } K \in \mathcal{T} \text{ and } \sum_{K\in\mathcal{T}} |K|\rho_K = M.$$

3.4.2 Estimates on the discrete solution

Lemma 3.4.1 *Let $\mathcal{T}$ be a triangulation of the computational domain Ω and Φ a nondecreasing function in $C^1(\mathbb{R}_*^+)$. Let $(\boldsymbol{u}, \rho) \in \boldsymbol{W}_h \times L_h$ satisfy the second equation of the scheme,* i.e. *Equation* (3.3.5b)*. Then, $\rho_K > 0$ for all $K \in \mathcal{T}$ and :*

$$\int_\Omega \Phi(\rho) \mathrm{div}_h(\boldsymbol{u})\, \mathrm{d}\boldsymbol{x} \leq 0.$$

Proof We first remark that ρ is solution of (3.4.1) with

$$a_{K,K} = h^\alpha |K| + \sum_{\sigma=K|L} \big(|\sigma| u_{K,\sigma}^+ + h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\rho_K| + |\rho_L|) \big),$$

$$a_{K,L} = -|\sigma| u_{K,\sigma}^- - h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\rho_K| + |\rho_L|) \text{ if } \sigma = K|L,$$

$$a_{K,L} = 0 \text{ if } K \text{ and } L \text{ do not share an interface.}$$

$$b_K = h^\alpha |K| \rho^\star.$$

Then, since $b_K > 0$ for all $K \in \mathcal{T}$, one has $\rho_K > 0$ for all $K \in \mathcal{T}$ (see Lemma 3.8.4).

Let the function $\psi \in C^1(\mathbb{R}_+^\star)$ satisfying $\psi'(s) = \frac{\Phi'(s)}{s}$ for all $s > 0$ (ψ is nondecreasing). Multiplying (3.3.5b) by $\psi(\rho_K)$ and summing over $K \in \mathcal{T}$ yields $T_1 + T_2 + T_3 = 0$ with :

$$T_1 = \sum_{K\in\mathcal{T}} \psi(\rho_K) \sum_{\sigma=K|L} |\sigma| \rho_\sigma \, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL},$$
$$T_2 = \sum_{K\in\mathcal{T}} h^\alpha \, |K| \, \psi(\rho_K) \, (\rho_K - \rho^*) \, ,$$
$$T_3 = \sum_{K\in\mathcal{T}} \psi(\rho_K) \sum_{\sigma=K|L} (h_K + h_L)^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K + \rho_L) \, (\rho_K - \rho_L) \, .$$

Let :

$$T_4 = \sum_{K\in\mathcal{T}} \int_K \Phi(\rho_K)\mathrm{div}(\boldsymbol{u}) = \sum_{\sigma=K|L} |\sigma|\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL}(\Phi(\rho_K) - \Phi(\rho_L))$$

We have : $T_4 = T_4 - T_1 - T_2 - T_3$

$$= \sum_{\sigma=K|L} |\sigma|\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL}[\Phi(\rho_K) - \Phi(\rho_L) - \rho_\sigma(\psi(\rho_K) - \psi(\rho_L))] - T_2 - T_3,$$

with $\rho_\sigma = \rho_K$ if $\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL} > 0$ and $\rho_\sigma = \rho_L$ if $\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL} < 0$.

The fact that ψ is nondecreasing yields :
$\star$ $T_2 \geq \sum_{K\in\mathcal{T}} h^\alpha |K| \psi(\rho^*) \, (\rho_K - \rho^*) = 0$,
$\star$ $T_3 = \sum_{\sigma=K|L} (h_K + h_L)^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K + \rho_L) \, (\rho_K - \rho_L) \, (\psi(\rho_K) - \psi(\rho_L)) \geq 0$.

For $\alpha > 0$, we define Φ_α on $\mathbb{R}_+^\star$ by $\Phi_\alpha(s) = \Phi(\alpha) - \Phi(s) - \alpha(\psi(\alpha) - \psi(s))$. Since Φ is nondecreasing (and $s\psi'(s) = \Phi'(s)$), one has $\Phi_\alpha(s) \leq 0$ for all $s \in \mathbb{R}_+^\star$. Then, thanks to the choice of ρ_σ, one has

$$\sum_{\sigma=K|L} |\sigma|\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL}[\Phi(\rho_K) - \Phi(\rho_L) - \rho_\sigma(\psi(\rho_K) - \psi(\rho_L))] \leq 0$$

which gives :

$$T_4 = \int_\Omega \Phi(\rho)\mathrm{div}_h(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \leq 0.$$

Proposition 3.4.2 *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \geq \theta_0$, where θ is defined by* (3.3.1)*. Let $(\boldsymbol{u}, p, \rho) \in \boldsymbol{W}_h \times L_h \times L_h$ be a solution of* (3.3.5)*. Then there exists C, only depending on the data of the problem Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, μ, φ, M and on θ_0, such that :*

$$\|\boldsymbol{u}\|_{1,b} \leq C, \; \|p\|_{\mathrm{L}^2(\Omega)} \leq C \text{ and } \|\rho\|_{\mathrm{L}^2(\Omega)} \leq C. \tag{3.4.2}$$

Proof
Let $(\boldsymbol{u}, p, \rho)$ be a solution of (3.3.5) .Taking $\boldsymbol{u}$ as test function in (3.3.5a) yields :

$$\mu \, \|\boldsymbol{u}\|_{1,b}^2 + \frac{\mu}{3} \int_\Omega \mathrm{div}_h^2(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} - \int_\Omega p \, \mathrm{div}_h(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \rho\boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}. \tag{3.4.3}$$

Using Lemma 3.4.1, a (well known) discrete Poincaré Inequality and the Hölder Inequality, one obtains the existence of C_1 only depending on Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$ such that

$$\|\boldsymbol{u}\|_{1,b} \leq C_1(1 + \|\rho\|_{\mathrm{L}^2(\Omega)}). \tag{3.4.4}$$

Since $p = \varphi(\rho)$, using (3.2.1), for all $\varepsilon > 0$ there exists C_ε (only depending on ε, φ and Ω) such that :

$$\|\rho\|_{\mathrm{L}^2(\Omega)} \leq C_\varepsilon + \varepsilon \|p\|_{\mathrm{L}^2(\Omega)}. \tag{3.4.5}$$

Then, with (3.4.4), for all $\varepsilon > 0$, there exists $\bar{C}_\varepsilon$, only depending on Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$, φ and ε such that

$$\|\boldsymbol{u}\|_{1,b} \leq \bar{C}_\varepsilon + \varepsilon \|p\|_{\mathrm{L}^2(\Omega)}. \tag{3.4.6}$$

We now use Lemma 3.8.2. There exists $\boldsymbol{w} \in \mathrm{H}_0^1(\Omega)^d$ such that $\mathrm{div}(\boldsymbol{w}) = p - m(p)$ a.e. in Ω and $\|\boldsymbol{w}\|_{\mathrm{H}^1(\Omega)^d} \leq c_2 \|p - m(p)\|_{\mathrm{L}^2(\Omega)}$ where c_2 only depends on Ω.

Taking $\boldsymbol{v} = r_h \boldsymbol{w}$ as test function in (3.3.5a) yields :

$$\begin{aligned}\int_\Omega p \,\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \mu \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3} \int_\Omega \mathrm{div}_h(\boldsymbol{u})\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Since $\int_\Omega \mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = 0$, this gives also

$$\begin{aligned}\int_\Omega [p - m(p)] \,\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \mu \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3} \int_\Omega \mathrm{div}_h(\boldsymbol{u})\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}\end{aligned}$$

and then

$$\begin{aligned}\int_\Omega [p - m(p)]^2 \,\mathrm{d}\boldsymbol{x} = \mu \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3} \int_\Omega \mathrm{div}_h(\boldsymbol{u})\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Using theorem 3.3.1, lemma 3.8.2 and the inequalities (3.4.5) and (3.4.6) we get for all $\varepsilon > 0$, the existence of D_ε, only depending on Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$, φ, θ_0 and ε such that

$$\|p - m(p)\|_{\mathrm{L}^2(\Omega)} \leq D_\varepsilon + \varepsilon \|p\|_{\mathrm{L}^2(\Omega)}.$$

In order to obtain an estimate on $\|p\|_{L^2}$, we now use the fact that $\int_\Omega \rho dx = M$ (and we will deduce an estimate on $\|p\|_{L^2}$ in term of Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$, θ_0, φ and M).

We first modify a little bit the function φ (which is only nondecreasing) in order to obtain a function $\bar{\varphi}$ continuous and one-to-one from $\mathbb{R}_+$ onto $\mathbb{R}_+$, so as to be able to use its inverse function. Let $s_0 > 0$ such that $\varphi(s_0) = 1$. We define the increasing function $\bar{\varphi}$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ by

$$\begin{aligned}\bar{\varphi}(s) &= \frac{s}{s_0} \text{ if } 0 \leq s \leq s_0, \\ \bar{\varphi}(s) &= s \max_{s \in [s_0, s]} \frac{\varphi(t)}{t} \text{ if } s_0 < s.\end{aligned}$$

The function $\bar{\varphi}$ is a continuous increasing and one-to-one function from $\mathbb{R}_+$ onto $\mathbb{R}_+$. Then, there exists ψ (continuous increasing and one-to-one) from $\mathbb{R}_+$ onto $\mathbb{R}_+$ such that

$$\psi(\bar{\varphi}(s)) = \bar{\varphi}(\psi(s)) = s \text{ for all } s \in \mathbb{R}_+.$$

Since $Im(\psi) = \mathbb{R}_+$, we have $\lim_{s\to+\infty} \psi(s) = +\infty$.

We also remark that for all $s \geq 0$ one has for $s \geq s_0$, $\bar{\varphi}(s) \geq \varphi(s)$ and then, a.e. in Ω,

$$\psi(p) = \psi(\varphi(\rho)) \leq \psi(\bar{\varphi}(\rho)) + \varphi(s_0) = \rho + 1.$$

This gives $\int_\Omega \psi(p)dx \leq M + |\Omega|$.

We now use Lemma 3.6.1. It gives the existence of $\bar{C}$, only depending on Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$, θ_0, φ and M such that

$$\|p\|_{L^2} \leq \bar{C}. \tag{3.4.7}$$

Using (3.4.7) in (3.4.4) we thus get the estimate on $\|\boldsymbol{u}\|_{1,b}$.

Finally, thanks to $p = \varphi(\rho)$ and (3.2.1), the estimate on ρ follows.

Lemma 3.4.3 *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \geq \theta_0$, where θ is defined by* (3.3.1). *Let $(\boldsymbol{u}, p, \rho) \in \boldsymbol{W}_h \times L_h \times L_h$ be a solution of* (3.3.5). *Then, there exists $\bar{C}$ only depending on Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, μ, φ, M and θ_0 such that*

$$h^\xi\, |\rho|^2_{\mathcal{T}} \leq \bar{C} \text{ and } E(\rho) \leq \bar{C}$$

where $E(\rho) = \displaystyle\sum_{\sigma=K|L} min(\frac{1}{\rho_K}, \frac{1}{\rho_L})(\rho_K - \rho_L)^2 |\sigma| |\boldsymbol{u}_{K,\sigma}|$.

Proof We recall that $\rho_K > 0$ for all $K \in \mathcal{T}$. Multiplying Equation (3.3.5b) by $\ln(\rho(K))$ and summing over $K \in \mathcal{T}$, we thus obtain :

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma}\rho_\sigma + \sum_{K\in\mathcal{T}} \ln(\rho_K) M_K + \sum_{K\in\mathcal{T}} \ln(\rho_K) T_K = 0,$$

with $\rho_\sigma = \rho_K$ if $\boldsymbol{u}_{K,\sigma} > 0$ and $\rho_\sigma = \rho_L$ if $\boldsymbol{u}_{K,\sigma} < 0$.

The fact that the function $s \in \mathbb{R}^*_+ \to \ln(s)$ is increasing yields :

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma}\rho_\sigma + \sum_{K\in\mathcal{T}} \ln(\rho_K) T_K \leq 0 \tag{3.4.8}$$

Reordering the summations in the second term yields :

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) T_K = \sum_{\sigma=K|L} h^\xi \frac{|\sigma|}{h_\sigma} (\rho_K + \rho_L) (\ln(\rho_K) - \ln(\rho_L)) (\rho_K - \rho_L).$$

Then, using the mean value theorem, for all $\sigma = K|L$ there exists $\tilde{\rho_\sigma}$ between ρ_K and ρ_L such that

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) T_K = \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \, \frac{(\rho_K + \rho_L)}{\tilde{\rho_\sigma}}, \; (\; \tilde{\rho_\sigma} \in (\rho_K, \rho_L))$$

and this gives $\displaystyle\sum_{K\in\mathcal{T}} \ln(\rho_K) T_K \geq \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 .$

Using this inequality in (3.4.8) we get

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_\sigma + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \leq 0$$

which can be rewritten as

$$\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_\sigma (\ln(\rho_K) - \ln(\rho_L)) + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \leq 0.$$

If $\sigma = K|L$, we now choose for K the cell satisfying $\boldsymbol{u}_{K,\sigma} \geq 0$. We thus obtain

$$\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_K (\ln(\rho_K) - \ln(\rho_L)) + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \leq 0.$$

Adding and substracting the quantity $\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} (\rho_K - \rho_L)$, we then get

$$\begin{aligned}\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} [\rho_K (\ln(\rho_K) - \ln(\rho_L)) - (\rho_K - \rho_L)] + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \\ \leq - \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} (\rho_K - \rho_L) = - \int_\Omega \rho \, \mathrm{div}_h \boldsymbol{u} \leq \; \|\rho\|_{\mathrm{L}^2(\Omega)} \; \|\boldsymbol{u}\|_{1,b} \, .\end{aligned}$$

Since we have $\|\rho\|_{\mathrm{L}^2(\Omega)} \leq C$ and $\|\boldsymbol{u}\|_{1,b} \leq C$ where C is given by Proposition 3.4.2, we obtain

$$\begin{aligned}\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} [\rho_K (\ln(\rho_K) - \ln(\rho_L)) - (\rho_K - \rho_L)] \\ + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \leq C^2.\end{aligned} \tag{3.4.9}$$

We now use Lemma 3.8.5 with $\psi(s) = \ln(s)$. We obtain the existence for $\sigma = KlL$ of $\tilde{\rho_\sigma}$ between ρ_K and ρ_L such that

$$\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} [\rho_K (\ln(\rho_K) - \ln(\rho_L)) - (\rho_K - \rho_L)] = \sum_{\sigma=K|L} \frac{1}{2} |\sigma| u_{K,\sigma} (\rho_K - \rho_L)^2 \tilde{\rho}_\sigma^{-1}.$$

Using this equality in (3.4.9), we get :

$$\underbrace{\sum_{\sigma\in\mathcal{E}_{int}} \frac{1}{2} |\sigma| u_{K,\sigma} (\rho_K - \rho_L)^2 \tilde{\rho}_\sigma^{-1}}_{S_1} + \underbrace{\sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2}_{S_2} \leq C^2.$$

This gives $S_1 \leq C^2$ and $S_2 \leq C^2$ and concludes the proof since $S_2 = h^\xi \, |\rho|_{\mathcal{T}}^2$ and $E(\rho) \leq S_1$.

3.4.3 Passing to the limit in the discrete problem

Theorem 3.4.4 *Let $\alpha > 0$ and $0 < \xi < 2$. Let a sequence of triangulations $(\mathcal{T}_n)_{n\in\mathbb{N}}$ of Ω be given. We assume that h_n (given by $\mathcal{T}_n$) tends to zero when $n \to \infty$. In addition, we assume that the sequence of discretizations is regular, in the sense that $\theta_n \geq \theta_0 > 0$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we denote by $W_h^{(n)}$ and L_{h_n} the discrete spaces (for velocity, pressure and density) associated to $\mathcal{T}_n$ and by $(\boldsymbol{u}_n, p_n, \rho_n) \in W_h^{(n)} \times L_{h_n} \times L_{h_n}$ a corresponding solution to the discrete problem* (3.3.5)*. Then, up to the extraction of a subsequence, when $n \to \infty$:*

1. *The sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ (strongly) converges in $\mathrm{L}^2(\Omega)^d$ to a limit $\boldsymbol{u} \in \mathrm{H}_0^1(\Omega)^d$ and $(p_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ converge weakly in $\mathrm{L}^2(\Omega)$ to p, ρ respectively ;*
2. *$(\boldsymbol{u}, p, \rho)$ is a solution to Problem* (3.2.4)*.*

Furthermore, if φ is increasing, the sequences $(p_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ converge in $L^p(\Omega)$ for $1 \leq p < 2$ (up to a subsequence).

Proof The proof is devided in four steps :

• **Step 1. Existence of a limit**

The convergence (up to the extraction of a subsequence) of the sequence $(\boldsymbol{u}_n, p_n, \rho_n)$ is a consequence of the uniform (with respect to n) estimates of Proposition 3.4.2 (applying Theorem 3.3.2 to each component of $\boldsymbol{u}_n$). Then (up to an extraction) the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ (strongly) converges in $\mathrm{L}^2(\Omega)^d$ to a limit $\boldsymbol{u} \in \mathrm{H}_0^1(\Omega)^d$ and $(p_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ converge weakly in $\mathrm{L}^2(\Omega)$ to p and ρ.

Since $\rho_n > 0$ and $\int_\Omega \rho_n \,\mathrm{d}\boldsymbol{x} = M$, we obtain, passing to the limit as $n \to \infty$, $\rho \geq 0$ a.e. and $\int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M$.

We now have to prove that (u, p) satisfies (3.2.4a) (this is proven in Step 2), that (u, ρ) statisfies (3.2.4b) (Step 3) and that $p = \varphi(\rho)$ a.e. (Step 4). Step 4 will also gives the strong convergence of ρ and p if φ is increasing.

• **Step 2. Passing to the limit in** (3.3.5a)

Let ψ be a function of $C_c^\infty(\Omega)^d$. We denote by ψ_n the interpolant of ψ in $W_h^{(n)}$, *i.e.* $\psi_n = r_{h_n}(\psi)$. Taking $v = \psi_n$ in ((3.3.5a)), we obtain :

$$\begin{aligned} \mu \int_\Omega \boldsymbol{\nabla}_{h_n}\boldsymbol{u}_n : \boldsymbol{\nabla}_{h_n}\psi_n \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3}\int_\Omega \mathrm{div}_{h_n}(\boldsymbol{u}_n)\mathrm{div}_{h_n}(\psi_n)\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega p_n \,\mathrm{div}_{h_n}(\psi_n)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\psi_n \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_n \boldsymbol{g}\cdot\psi_n\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.4.10}$$

We now write $\int_\Omega \boldsymbol{\nabla}_{h_n}\boldsymbol{u}_n : \boldsymbol{\nabla}_{h_n}\psi_n \,\mathrm{d}\boldsymbol{x} = T_1 + T_2$ with

$$T_1 = \int_\Omega \boldsymbol{\nabla}_{h_n}\boldsymbol{u}_n : \boldsymbol{\nabla}_{h_n}(\psi_n - \psi)\,\mathrm{d}\boldsymbol{x} \text{ and } T_2 = \int_\Omega \boldsymbol{\nabla}_{h_n}\boldsymbol{u}_n : \boldsymbol{\nabla}_{h_n}\psi\,\mathrm{d}\boldsymbol{x}.$$

Using the third property of the interpolation operator given in theorem 3.3.1, we get, with $c(\theta_0)$ only depending on Ω and θ_0,

$$|T_1| \leq \|\boldsymbol{u}_n\|_{1,b}\ \|(\psi_n - \psi)\|_{1,b} \leq c(\theta_0) h_n \|\boldsymbol{u}_n\|_{1,b}\, |\psi|_{\mathrm{H}^2(\Omega)}$$

and thus T_1 tends to zero as n tends to $+\infty$. Integrating by parts over each control volume, the term T_2 reads :

$$T_2 = -\int_\Omega \boldsymbol{u}_n \cdot \Delta\psi \,\mathrm{d}\boldsymbol{x} + \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma [\boldsymbol{u}_n]_\sigma : \boldsymbol{\nabla}\psi \,\mathrm{d}\gamma,$$

where $[\boldsymbol{u}_n]_\sigma = (\boldsymbol{u}_K \otimes \boldsymbol{n}_K + \boldsymbol{u}_L \otimes \boldsymbol{n}_L)$ if $\sigma = K|L$ (for all $K \in \mathcal{T}_n$, $\boldsymbol{u}_K$ is the value of $\boldsymbol{u}_n$ in K, and $\boldsymbol{n}_K$ is the normal vector to ∂K exterior to K). We omit the dependance of $\mathcal{E}_{\mathrm{int}}$ with respect to n. Noticing that $\boldsymbol{n}_L = -\boldsymbol{n}_K$ and applying Lemma 2.4 in [11], we get, again with $c(\theta_0)$ only depending on Ω and θ_0,

$$\Big| \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma [\boldsymbol{u}_n] : \boldsymbol{\nabla}\psi \; n_\sigma \,\mathrm{d}\gamma \Big| \le c(\theta_0) h_n \, \|\boldsymbol{u}_n\|_{1,b} \, |\psi|_{\mathrm{H}^2(\Omega)}$$

and thus tends to zero as n tends to $+\infty$. On the other hand we have :

$$-\int_\Omega \boldsymbol{u}_n \cdot \Delta\psi \,\mathrm{d}\boldsymbol{x} \to -\int_\Omega \boldsymbol{u} \cdot \Delta\psi \,\mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty$$
$$= \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla}\psi \,\mathrm{d}\boldsymbol{x} \text{ since } \boldsymbol{u} \in \mathrm{H}^1_0(\Omega).$$

Then, the first term of the Left Hand Side of (3.4.10) converges to $\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla}\psi \,\mathrm{d}\boldsymbol{x}$ as $n \to \infty$. For the second term of (3.4.10), using the first property of the interpolation operator in theorem 3.3.1, we get, with $[\boldsymbol{u}_n \cdot \boldsymbol{n}]_\sigma = \boldsymbol{u}_K \cdot \boldsymbol{n}_K + \boldsymbol{u}_L \cdot \boldsymbol{n}_L$,

$$\begin{aligned}
\int_\Omega \mathrm{div}_{h_n}(\boldsymbol{u}_n)\mathrm{div}_{h_n}(\psi_n) \,\mathrm{d}\boldsymbol{x} &= \int_\Omega \mathrm{div}_{h_n}(\boldsymbol{u}_n)\mathrm{div}(\psi) \,\mathrm{d}\boldsymbol{x} \\
= \sum_{K\in\mathcal{T}_n} \sum_{i\le d} \sum_{j\le d} \int_K (\boldsymbol{u}_n)_i \frac{\partial^2 \psi_j}{\partial x_i \partial x_j} \,\mathrm{d}\boldsymbol{x} &+ \sum_{K\in\mathcal{T}_n} \int_{\partial K} \boldsymbol{u}_n \mathrm{div}\psi . n_K \,\mathrm{d}\gamma \\
= \sum_{K\in\mathcal{T}_n} \sum_{i\le d} \sum_{j\le d} \int_K (\boldsymbol{u}_n)_i \frac{\partial^2 \psi_j}{\partial x_i \partial x_j} \,\mathrm{d}\boldsymbol{x} &+ \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma [\boldsymbol{u}_n \cdot \boldsymbol{n}]_\sigma \mathrm{div}\psi \,\mathrm{d}\gamma \\
&= T_{2,1} + T_{2,2}.
\end{aligned}$$

Applying Lemma 2.4 in [11], we get, with $c(\theta_0)$ only depending on Ω and θ_0,

$$|T_{2,2}| = \Big| \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma [\boldsymbol{u}_n \cdot \boldsymbol{n}]_\sigma \mathrm{div}\psi \,\mathrm{d}\gamma \Big| \le c(\theta_0) h_n \, \|\boldsymbol{u}_n\|_{1,b} \, |\mathrm{div}\psi|_{\mathrm{H}^1(\Omega)}$$

and thus $T_{2,2}$ tends to zero as n tends to $+\infty$. Then, the second term of (3.4.10) has the same limit as $T_{2,1}$ and this limit is $\int_\Omega \mathrm{div}\boldsymbol{u} \; \mathrm{div}\psi \,\mathrm{d}\boldsymbol{x}$.

For the third term of (3.4.10), we use, once again, Theorem 3.3.1 which yields :

$$\int_\Omega p_n \; \mathrm{div}_{h_n}(\psi_n) \,\mathrm{d}\boldsymbol{x} = \int_\Omega p_n \; \mathrm{div}(\psi) \,\mathrm{d}\boldsymbol{x} \to \int_\Omega p \; \mathrm{div}(\psi) \,\mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty.$$

We now consider the Right Hand Side of (3.4.10). Since $\psi_n \to \psi$ in $L^2(\Omega)^d$ we obtain

$$\int_\Omega \boldsymbol{f} \cdot \psi_n \,\mathrm{d}\boldsymbol{x} \to \int_\Omega \boldsymbol{f} \cdot \psi \,\mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty.$$

For the last term of (3.4.10), we use, once again, the $(L^2)^d$ convergence of ψ_n to ψ and we use the weak-L^2 convergence of ρ_n to ρ. We obtain

$$\int_\Omega \rho_n \boldsymbol{g} \cdot \psi_n \,\mathrm{d}\boldsymbol{x} \to \int_\Omega \rho \boldsymbol{g} \cdot \psi \,\mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty.$$

Finally, we can pass to limit in (3.4.10) as $n \to \infty$ and we get (3.2.4a) for all $\psi \in C_c^\infty(\Omega)^d$ (and then, by density, for all $\psi \in H_0^1(\Omega)^d$), namely :

$$\begin{aligned} \mu \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla}\psi \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3} \int_\Omega \mathrm{div}(\boldsymbol{u})\mathrm{div}(\psi) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\mathrm{div}(\psi) \,\mathrm{d}\boldsymbol{x} \\ = \int_\Omega \boldsymbol{f} \cdot \psi \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{g} \cdot \psi \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

• **Step 3. Passing to the limit in** (3.3.5b)

Let ψ be a function of $C_c^\infty(\Omega)^d$.Multiplying (3.3.5b) by $\psi_K = \psi(x_K)$ and summing over $K \in \mathcal{T}_n$ we obtain :

$$\begin{aligned} T_1 + T_2 + T_3 = \sum_{K \in \mathcal{T}_n} \psi_K \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_\sigma + \sum_{K \in \mathcal{T}_n} h_n^\alpha |K| \psi_K (\rho_K - \rho^*) \\ + \sum_{K \in \mathcal{T}_n} \psi_K \sum_{\sigma \in \mathcal{E}(K)} h_n^\xi \frac{|\sigma|}{h_\sigma} (\rho_K + \rho_L)(\rho_K - \rho_L) = 0. \end{aligned} \tag{3.4.11}$$

The first term T_1 reads, with $\psi_\sigma = \psi(x_\sigma)$,

$$\begin{aligned} T_1 = \sum_{K \in \mathcal{T}_n} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_\sigma \psi_K = \sum_{K \in \mathcal{T}_n} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_\sigma (\psi_K - \psi_\sigma) \\ = \sum_{K \in \mathcal{T}_n} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_K (\psi_K - \psi_\sigma) + R_1 \\ \text{with } R_1 = - \sum_{K \in \mathcal{T}_n} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma} (\rho_K - \rho_\sigma)(\psi_K - \psi_\sigma). \end{aligned}$$

Then,

$$\begin{aligned} T_1 = \sum_{K \in \mathcal{T}_n} \rho_K \psi_K \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma} - \sum_{K \in \mathcal{T}_n} \rho_K \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma} \psi_\sigma + R_1 \\ = \sum_{K \in \mathcal{T}_n} \rho_K \int_K \psi \mathrm{div} u_n \,\mathrm{d}\boldsymbol{x} - \sum_{K \in \mathcal{T}_n} \rho_K \int_K \mathrm{div}(\psi u_n) \,\mathrm{d}\boldsymbol{x} + R_1 + R_2 + R_3, \\ \text{with } R_2 = - \sum_{K \in \mathcal{T}_n} \rho_K \int_K (\psi - \psi_K) \mathrm{div} \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} \end{aligned}$$

$$\text{and } R_3 = \sum_{K\in\mathcal{T}_n} \rho_K \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma (\psi - \psi_\sigma)\boldsymbol{u}_n \cdot \boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma.$$

Therefore, $T_1 = -\displaystyle\int_\Omega \rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla}\psi + R_1 + R_2 + R_3$.

Let us now prove that the terms $R_1, R_2, R_3 \to 0$ as n $\to +\infty$. We begin with R_1. One has, with $C_\psi = \||\boldsymbol{\nabla}\psi|\|_{L^\infty(\Omega)}$,

$$\begin{aligned} |R_1| &= |\sum_{K\in\mathcal{T}_n}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\boldsymbol{u}_{K,\sigma}(\rho_K-\rho_\sigma)(\psi_K-\psi_\sigma)| \\ &\le C_\psi \sum_{\sigma=K|L} (h_K+h_L)|\rho_K-\rho_L||\sigma||\boldsymbol{u}_{K,\sigma}|. \end{aligned}$$

This gives, with the Cauchy-Schwarz inequality,

$$|R_1| \le C_\psi E(\rho_n)\Big(\sum_{\sigma=K|L} \frac{(h_K+h_L)^2}{min(\frac{1}{\rho_K},\frac{1}{\rho_L})}|\sigma||\boldsymbol{u}_{K,\sigma}|\Big)^{\frac12}.$$

Then,

$$|R_1| \le C_\psi E(\rho_n)(\underbrace{\sum_{\sigma=K|L}(h_K+h_L)^2(\rho_K+\rho_L)|\sigma||\boldsymbol{u}_{K,\sigma}|}_{S2})^{\frac12}.$$

Using again the Cauchy Schwarz inequality we thus obtain :

$$S_2 \le (\sum_{\sigma=K|L}(h_K+h_L)|\sigma|(\rho_K+\rho_L)^2)^{1/2}(\sum_{\sigma=K|L}(h_K+h_L)^3|\sigma||\boldsymbol{u}_{K,\sigma}|^2)^{1/2}$$

The properties of the scheme given in section 2 and Hölder's Inequality yields, with $C_1(\theta_0)$ and $C_2(\theta_0)$ only depending on Ω and θ_0,

$$S_2 \le C_1(\theta_0)(\sum_{\sigma=K|L}(|K|+|L|)(\rho_K+\rho_L)^2)^{1/2}(\sum_{\sigma=K|L}(h_K+h_L)^3\|\boldsymbol{u}_n\|^2_{L^2(\sigma)})^{1/2}$$

$$\le C_2(\theta_0)(\sum_{K\in\mathcal{T}_n}|K|\rho_K^2)^{1/2}(\sum_{\sigma=K|L}h_\sigma^3\|\boldsymbol{u}_n\|^2_{L^2(\sigma)})^{1/2}.$$

The estimate on ρ_n in $L^2(\Omega)$ gives the existence of C_3, only depending on the L^2-bound on ρ_n and on $C_2(\theta_0)$ such that :

$$S_2 \le C_3\Big(\sum_{\sigma=K|L}h_\sigma^3\|\boldsymbol{u}_n\|^2_{L^2(\sigma)}\Big)^{1/2}.$$

By Lemma 2.3 in [11], we have :

$$\|\boldsymbol{u}_n\|_{L^2(\sigma)} \le (d\frac{|\sigma|}{|K|})^{1/2}(\|\boldsymbol{u}_n\|_{L^2(K)} + h_K\|\boldsymbol{\nabla}\boldsymbol{u}_n\|_{L^2(K)}).$$

We thus obtain, with some C_4 and C_5 only depending on the L^2-bound on ρ_n, Ω and θ_0,

$$S_2 \leq C_4\Big(\sum_{K\in\mathcal{T}_n} h_K^2(\|\boldsymbol{u}_n\|^2_{L^2(K)} + h_K^2\|\boldsymbol{\nabla}\boldsymbol{u}_n\|^2_{L^2(K)})\Big)^{1/2} \leq C_5 h_n(\|\boldsymbol{u}_n\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_n\|^2_{1,b})^{1/2}.$$

Finally, thanks to the bound on u_n (Proposition 3.4.2) we get $\lim_{n\to\infty} S_2 = 0$ and thanks to the bound on $E(\rho_n)$ (Lemma 3.4.3) we conclude that $\lim_{n\to\infty} R_1 = 0$.

We now come to R_2. One has

$$|R_2| \leq C_\psi h_n \,\|\rho_n\|_{\mathrm{L}^2(\Omega)}\ \|\mathrm{div}_{h_n}(\boldsymbol{u}_n)\|_{\mathrm{L}^2(\Omega)} \ \leq C_\psi h_n \,\|\rho_n\|_{\mathrm{L}^2(\Omega)}\ \|\boldsymbol{u}_n\|_{1,b}\,,$$

which tends to 0 as $n \to +\infty$.

It remains to treat R_3. One has

$$\begin{aligned}
|R_3| &= |\sum_{K\in\mathcal{T}_n} \rho_K \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma (\psi - \psi_\sigma)\boldsymbol{u}_n\cdot\boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma| \\
&= |\sum_{\sigma=K|L} (\rho_K - \rho_L)\int_\sigma (\psi-\psi_\sigma)\boldsymbol{u}_n\cdot\boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma| \leq C_\psi \sum_{\sigma=K|L} |\rho_K-\rho_L| \int_\sigma h_\sigma|\boldsymbol{u}_n|\,\mathrm{d}\gamma.
\end{aligned}$$

Using the same arguments as for the first term R_1 (bound on $\boldsymbol{u}_n$ and bound on $E(\rho_n)$) we get a bound on $R_3 h_n^{-1/2}$ which gives $\lim_{n\to\infty} R_3 = 0$.

Finally, since $\lim_{n\to\infty} R_i = 0$ for $i = 1, 2, 3$, one has

$$\lim_{n\to\infty} T_1 = -\lim_{n\to\infty}\int_\Omega \rho_n u_n\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x}.$$

Using the $L^2(\Omega)$ convergence of $\boldsymbol{u}_n$ and the $L^2(\Omega)$-weak convergence of ρ_n, we conclude that

$$\lim_{n\to\infty} T_1 = -\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x}.$$

We now prove that T_2 and T_3 tend to 0 as $n\to\infty$. We remark that

$$|\ T_2\ | = |\sum_{K\in\mathcal{T}_n} h_K^\alpha\,|K|\ (\rho_K-\rho^*)\,\psi_K\ | \leq h_n^\alpha 2M\,\|\psi\|_{L^\infty(\Omega)}\ \to 0 \text{ as } n\to+\infty$$

and

$$\begin{aligned}
|\ T_3\ | &= |\sum_{K\in\mathcal{T}_n}\sum_{\sigma\in\mathcal{E}(K)} h_n^\xi\,\frac{|\sigma|}{h_\sigma}\ (\rho_K+\rho_L)\ (\rho_K-\rho_L)\,\psi_K\ | \\
&= |\ h_n^\xi \sum_{\sigma=K|L} \frac{|\sigma|}{h_\sigma}\ (\rho_K+\rho_L)\ (\rho_K-\rho_L)\,(\psi_K-\psi_L)\ | \\
&\leq C_\psi\, h_n^\xi \sum_{\sigma=K|L} (h_K+h_L)\ \frac{|\sigma|}{h_\sigma}\ (\rho_K+\rho_L)\ |\ \rho_K-\rho_L\ |\ .
\end{aligned}$$

We now use the Cauchy-Schwarz Inequality to obtain, with C_1 only depending on ψ and the bound on $h_n^\xi|\rho_n|^2$ given by Lemma 3.4.3,

$$|T_3| \le C_\psi\, h_n^\xi \mid \rho_n \mid_{\mathcal{T}} \Big(\sum_{\sigma=K|L} (h_K+h_L)^2 \, \frac{|\sigma|}{h_\sigma} \, (\rho_K+\rho_L)^2 \Big)^{\frac12}$$
$$\le C_1\, h_n^{\xi/2} \Big(\sum_{\sigma=K|L} (h_K+h_L)^2 \, \frac{|\sigma|}{h_\sigma} \, (\rho_K+\rho_L)^2 \Big)^{\frac12}.$$

The properties of the mesh given in section 2 yield the existence of $c(\theta_0)$ only depending on Ω and θ_0 such that

$$\frac{|\sigma|}{h_\sigma} \le c(\theta_0) \frac{|K|+|L|}{(h_K+h_L)^2}.$$

We thus obtain $\mid T_3 \mid \le C_1\sqrt{c(\theta_0)}\, h_n^{\xi/2}(\sum_{\sigma=K|L}(|K|+|L|)\,(\rho_K+\rho_L)^2)^{\frac12}$. Thanks to the L^2-estimate on ρ_n, we then conclude that $\lim_{n\to\infty} T_3 = 0$.

Finally, we can pass to the limit in (3.4.11) as $n \to \infty$ and we obtain (3.2.4b) for all $\psi \in C_c^\infty(\Omega)$. This gives also (3.2.4b) for all $\psi \in W^{1,\infty}(\Omega)$ thanks to Lemma 3.7.6 (since $u \in H_0^1(\Omega)$ and $\rho \in L^2(\Omega)$).

• **Step 4. Passing to the limit in the Equation Of State**

In order to conclude the proof of Theorem 3.4.4, it remains to prove that the Equation Of State is satisfied, that is $p = \varphi(\rho)$ a.e. in Ω. This is a tricky part of the proof.

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence such that $q_n \in L_{h_n}$ for all $n \in \mathbb{N}$. We assume that the sequence $(q_n)_{n\in\mathbb{N}}$ weakly converges in $\mathrm{L}^2(\Omega)$ to $q \in \mathrm{L}^2(\Omega)$ and satisfies

$$|q_n|_{\mathcal{T}} \le c\, h_n^{-\eta},$$

where c is a positive real number and η is such that $\eta < 1$. Then one has :

$$\forall \psi \in \mathrm{C}_c^\infty(\Omega), \quad \lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_{h_n}(\boldsymbol{u}_n) - p_n\big)\, q_n \psi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\, q\psi \,\mathrm{d}\boldsymbol{x}. \tag{3.4.12}$$

This result is proven in [6], Proposition 5.9. Indeed, in Proposition 5.9 of [6] the hypothesis on ρ is $\rho \in L^{2\gamma}(\Omega)$, $\gamma > 1$, and the sequence $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ weakly in $L^{2\gamma}(\Omega)$, but the proof given in [6] is also true for $\gamma = 1$.

Taking $q_n = \rho_n$ in (3.4.12) (which is possible with $\eta = \xi/2$, thanks to Lemma 3.4.3), one obtains

$$\forall \psi \in \mathrm{C}_c^\infty(\Omega), \quad \lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_{h_n}(\boldsymbol{u}_n) - p_n\big)\, \rho_n \psi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\, \rho\psi \,\mathrm{d}\boldsymbol{x}. \tag{3.4.13}$$

We now want to prove (3.4.13) with $\psi = 1$ a.e. on Ω. This is possible, thanks to Lemma 3.8.1, if the sequence $((\mathrm{div}_{h_n}\boldsymbol{u}_n - p_n)\rho_n)_{n\in\mathbb{N}}$ is equi-integrable. The condition (3.2.1) on φ, and the L^2-bound on $\mathrm{div}_{h_n}\boldsymbol{u}_n$ and on p_n will give this equi-integrability.

Let $a > 0$ and $b > 0$ given by (3.2.1). One has a.e. on Ω,

$$a\rho_n \leq \varphi(\rho_n) + b = p_n + b,$$

so that

$$\rho_n^2 \leq \frac{2p_n^2}{a^2} + \frac{2b^2}{a^2}.$$

If C is a bound for the L^2-norm of p_n (such a bound is given by Proposition 3.4.2), one obtains for any borelian subset A of Ω,

$$\int_A \rho_n^2 dx \leq \frac{2C^2}{a^2} + \frac{2b^2}{a^2}|A|.$$

Let $\varepsilon > 0$, we then take $a^2 = 2C^2/\varepsilon$ which yields :

$$\int_A \rho_n^2 dx \leq \varepsilon + \frac{2b^2}{a^2}|A|.$$

and then, with $\delta = \dfrac{\varepsilon a^2}{2b^2}$,

$$|A| \leq \delta \Rightarrow \int_A \rho_n^2 dx \leq 2\varepsilon.$$

This proves the equi-integrability of the sequence $(\rho_n^2)_{n\in\mathbb{N}}$. Since the sequence $((\mathrm{div}_{h_n}\boldsymbol{u}_n - p_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$, we then easily conclude (with the Cauchy-Schwarz inequality) that the sequence $((\mathrm{div}_{h_n}\boldsymbol{u}_n - p_n)\rho_n)_{n\in\mathbb{N}}$ is equi-integrable. Thus Lemma 3.8.1 yields the conclusion, namely (3.4.13) is true for $\psi = 1$ a.e. on Ω :

$$\lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_{h_n}(\boldsymbol{u}_n) - p_n\big)\, \rho_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\, \rho \,\mathrm{d}\boldsymbol{x}. \tag{3.4.14}$$

We now want to get rid of $\int_\Omega \rho \mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}$ and $\int_\Omega \rho_n \mathrm{div}(\boldsymbol{u}_n)\,\mathrm{d}\boldsymbol{x}$ in (3.4.14).

Since $\rho \in L^2(\Omega)$, $\rho \geq 0$ a.e. in Ω, $u \in H_0^1(\Omega)^d$ and (ρ, u) satisfies (3.2.4b), we can use Lemma 3.7.1. It gives

$$\int_\Omega \rho\ \mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = 0. \tag{3.4.15}$$

Then, using (3.4.15) in (3.4.14) we get :

$$\lim_{n\to\infty} \int_\Omega \big(p_n - \mathrm{div}_{h_n}(\boldsymbol{u}_n)\big)\, \rho_n \,\mathrm{d}\boldsymbol{x} - \int_\Omega p\, \rho \,\mathrm{d}\boldsymbol{x} = 0.$$

By Lemma 3.4.1 we also have $\int_\Omega \rho_n \mathrm{div}_{h_n}(\boldsymbol{u}_n)\ \mathrm{d}\boldsymbol{x} \leq 0$. Hence :

$$\limsup_{n\to\infty} \int_\Omega p_n\, \rho_n \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega p\, \rho \,\mathrm{d}\boldsymbol{x}. \tag{3.4.16}$$

To conclude the proof of $p = \varphi(\rho)$, we will now use the so called Minty trick. Let $\bar\rho \in L^2(\Omega)$ such that $\varphi(\bar\rho) \in L^2(\Omega)$. We define for $n \in \mathbb{N}$ the function G_n by

$$G_n = (\varphi(\rho_n) - \varphi(\bar\rho))(\rho_n - \bar\rho) = (p_n - \varphi(\bar\rho))(\rho_n - \bar\rho).$$

One has $G_n \in L^1(\Omega)$, $G_n \geq 0$ a.e. in Ω (since φ is nondecreasing) and

$$0 \leq \int_\Omega G_n \, dx = \int_\Omega (p_n\rho_n - p_n\bar\rho - \varphi(\bar\rho)\rho_n + \varphi(\bar\rho)\bar\rho) \, \mathrm{d}\boldsymbol{x}. \tag{3.4.17}$$

Using (3.4.16) and the weak convergences of p_n to p and ρ_n to ρ in $L^2(\Omega)$, we obtain :

$$0 \leq \limsup_{n\to\infty} \int_\Omega G_n \, dx \leq \int_\Omega (p - \varphi(\bar\rho))(\rho - \bar\rho) \, \mathrm{d}\boldsymbol{x}.$$

We have thus proven that for all $\bar\rho \in L^2(\Omega)$ such that $\varphi(\bar\rho) \in L^2(\Omega)$ one has

$$\int_\Omega (p - \varphi(\bar\rho))(\rho - \bar\rho) \, \mathrm{d}\boldsymbol{x} \geq 0. \tag{3.4.18}$$

We now have to choose $\bar\rho$ conveniently to deduce $p = \varphi(\rho)$ a.e. on Ω from (3.5.11). The idea of the Minty trick is to take $\bar\rho = \rho + (1/k)\psi$ with $\psi \in C_c^\infty(\Omega)$, $k \in \mathbb{N}^\star$ and to let k goes to $+\infty$. Unfortunately, $\varphi(\rho + (1/k)\psi)$ is not necessarily in $L^2(\Omega)$. then, such a choice for $\bar\rho$ is not possible. We will use here (and only here) the convexity of φ. Since $(\rho_n)_n$ weakly converges in $L^2(\Omega)$ to ρ and since the sequence $(\varphi(\rho_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, we deduce, using the convexity of φ, that $\varphi(\rho) \in L^2(\Omega)$. This is proven in Lemma 3.7.8. This allows us a convenient choice for $\bar\rho$.

Let $\psi \in C_c^\infty(\Omega, \mathbb{R})$. For $k, m \in \mathbb{N}^\star$, we set

$$\rho_{k,m} = \rho + \frac{1}{k}\psi 1_{\rho\leq m}.$$

Since $\rho \in L^2(\Omega)$, one has $\rho_{k,m} \in L^2(\Omega)$. Using the fact that φ is nondecreasing (and nonnegative), we have, with $M = \|\psi\|_{L^\infty(\Omega)}$,

$$\varphi(\rho_{k,m}) \leq \varphi(\rho) + \varphi(m + M),$$

so that $\varphi(\rho_{k,m}) \in L^2(\Omega)$ (since $\varphi(\rho) \in L^2(\Omega)$). Then, since $\rho_{k,m}$ and $\varphi(\rho_{k,m})$ belong to $L^2(\Omega)$, we can choose $\bar\rho = \rho_{k,m}$ in (3.5.11). We obtain

$$\int_\Omega (p - \varphi(\rho + \frac{1}{k}\psi 1_{\rho\leq m}))\psi 1_{\rho\leq m} \leq 0.$$

Fixing m in $\mathbb{N}^\star$, we use the Dominated Convergence theorem on the sequence $(g_k)_{k\in\mathbb{N}^\star}$ with $g_k = (p - \varphi(\rho + \frac{1}{k}\psi 1_{\rho\leq m}))\psi 1_{\rho\leq m}$. Indeed, the continuity of φ gives $g_k \to (p - \varphi(\rho))\psi 1_{\rho\leq m}$ a.e. in Ω. Furthermore, since φ is nondecreasing, one has, for all $n \in \mathbb{N}^\star$,

$$|g_k| \leq H = [p + \varphi(\rho) + \varphi(m + M)]|\psi| \text{ a.e. in } \Omega,$$

and $H \in L^1(\Omega)$. Then, the Dominated Convergence theorem yields

$$\int_\Omega (p - \varphi(\rho))\psi 1_{\rho \le m} \le 0.$$

Changing ψ in $-\psi$, we conclude that $\int_\Omega (p - \varphi(\rho))\psi 1_{\rho \le m} = 0$ for all $\psi \in C_c^\infty(\Omega, \mathbb{R})$.

Once again by the Dominated Convergence Theorem, as m to $+\infty$ we get : $\int_\Omega (p - \varphi(\rho))\psi = 0$ for all $\psi \in C_c^\infty(\Omega)$ This gives $p = \varphi(\rho)$ a.e. in Ω.

The proof of Theorem 3.4.4 is now complete.

Remark 3.4.5 *The hypothesis of convexity of the function φ is only used to get that the four terms of the Right Hand Side of* (3.4.17) *are in $L^1(\Omega)$. If the hypothesis of convexity for φ is replaced by the hypothesis* (3.2.3)*, the proof is a little simpler, more details in this case are given in section 3.5.*

In both cases (φ convex or φ satisfies (3.2.3)*), if φ is increasing, we can obtain a strong convergence of ρ_n and p_n, as in [6]. We take directly $\bar{\rho} = \rho$ in the definition of G_n. We obtain that $G_n = (\varphi(\rho_n) - \varphi(\rho))(\rho_n - \rho) \to 0$ in $L^1(\Omega)$ as $n \to \infty$. Then, up to a subsequence, one has $G_n \to 0$ a.e. in Ω. Since φ is increasing, we finally deduce that $\rho_n \to \rho$ a.e.. This yields also $\rho_n \to \rho$ in $L^q(\Omega)$ for all $q \in [1, 2[$, $p = \varphi(\rho)$ a.e. in Ω and $p_n \to p$ in $L^q(\Omega)$ for all $q \in [1, 2[$.*

3.5 Sationnary compressible Stokes problem without gravity effects and φ non-convex

In this section, we propose a discretization for problem (3.2.2) with an equation of state of the form $p = \varphi(\rho)$ and φ satisfying the following condition $\exists a, \tilde{a}, b, \tilde{b} > 0$ and $\gamma > 1$ such that :

$$\forall s \in \mathbb{R}_+,\ as^\gamma - b \le \varphi(s) \le \tilde{a}s^{2\gamma-1} + \tilde{b}. \tag{3.5.1}$$

The discretization is the same given in section 3.3 , with a small change in the stabilization term T_K in the discrete mass balance equation. We prove estimates for the discrete solution, then the convergence of the scheme to a solution of the continuous problem is established.

3.5.1 The numerical scheme

Let α, ζ and ξ be given, with $\alpha > 0$, $0 < \xi < 2$ and $\zeta = \max(0, 2 - \gamma)$.We consider the following numerical scheme :

$$\forall \boldsymbol{v} \in \boldsymbol{W}_h, \mu \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3} \int_\Omega \mathrm{div}_h(\boldsymbol{u})\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \tag{3.5.2a}$$

$$\forall K \in \mathcal{T}, \qquad \sum_{\sigma=K|L} \mathrm{v}^+_{\sigma,K}\, \rho_K - \mathrm{v}^-_{\sigma,K}\, \rho_L + M_K + T_K = 0, \tag{3.5.2b}$$

$$\forall K \in \mathcal{T}, \qquad p_K = \varphi(\rho_K), \tag{3.5.2c}$$

the terms M_K and T_K read :

$$M_K = h^\alpha \, |K| \, (\rho_K - \rho^*) \,, \tag{3.5.3a}$$

$$T_K = \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\rho_K| + |\rho_L|)^\zeta \, (\rho_K - \rho_L) \,. \tag{3.5.3b}$$

Remark 3.5.1 *Problem(3.5.2) admits a solution, the proof is the same to that given in section 3.4.1.*

3.5.2 Convergence of approximate solutions

Estimates on the discrete solution

Theorem 3.5.2 *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \geq \theta_0$, where θ is defined by* (3.3.1)*. Let $(\boldsymbol{u}, p, \rho) \in \boldsymbol{W}_h \times L_h \times L_h$ a solution of* (3.5.2)*, then there exist C1, C2, C3 and C4, only depending on the data of the problem Ω, $\boldsymbol{f}$, M and on θ_0 such that :*

$$\|\boldsymbol{u}\|_{1,b} \leq C1, \; \|p\|_{\mathrm{L}^2(\Omega)} \leq C2, \; \|\rho\|_{\mathrm{L}^{2\gamma}(\Omega)} \leq C3 \text{ and } h^{\xi/2}\, |\rho|_{\mathcal{T}} \leq C4 \tag{3.5.4}$$

Proof

Let $(\boldsymbol{u}, p, \rho)$ be a solution of (3.5.2) . Taking $\boldsymbol{u}$ as test function in (3.5.2a) yields :

$$\mu \, \|\boldsymbol{u}\|^2_{1,b} \, + \frac{\mu}{3} \int_\Omega \mathrm{div}^2_h(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\mathrm{div}(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}. \tag{3.5.5}$$

Using Lemma 3.4.1, a discrete Poincaré Inequality and the Hölder inequality, yields the existence of C_1 only depending on $\Omega, \boldsymbol{f}, \mu$ and θ_0 such that $\|\boldsymbol{u}\|_{1,b} \leq C_1$. Using the *inf-sup* stability of the discretization, we hence get from (3.5.5) a control of $\|p - m(p)\|_{\mathrm{L}^2(\Omega)}$ (where $m(p)$ stands for the mean value of p over Ω).

In order to obtain an estimate on p, we give here a proof using the condition (3.5.1) , an other proof is possible using lemma 3.6.1. We set (for simplicity)$\varphi(s) = s + \varphi(0)$ for $s < 0$

and we define the function Φ from $\mathbb{R}$ to $\mathbb{R}$ by $\Phi(s) = \inf\{t \in \mathbb{R}_+;\ s = 3\varphi(t)\}$. The function Φ satisfies the following properties :

$$s = 3\varphi(t) \Rightarrow \Phi(s) \leq t, \tag{3.5.6a}$$

$$s = 3\varphi(\Phi(s)), \tag{3.5.6b}$$

$$\Phi(s) \to +\infty,\ \text{as } s \to +\infty, \tag{3.5.6c}$$

$$\Phi \text{ is nondecreasing.} \tag{3.5.6d}$$

For all $x \in \Omega$ one has

$$m(p) \leq |m(p) - p(x)| + |p(x)| \leq |m(p) - p(x)| + 2|\varphi(0)| + p(x).$$

Then, using (3.5.6d),

$$\Phi(m(p)) \leq \Phi(3|m(p) - p(x)|) + \Phi(6|\varphi(0)|) + \Phi(3p(x)).$$

Since $3p(x) = 3\varphi(\rho(x))$, (3.5.6a) gives

$$\Phi(m(p)) \leq \Phi(3|m(p) - p(x)|) + \Phi(6|\varphi(0)|) + \rho(x).$$

By summing equation (3.5.2b) for $K \in \mathcal{T}$, we obtain that the integral of ρ over Ω is M, which yields :

$$\int_\Omega \Phi(m(p))dx \leq \int_\Omega \Phi(3|m(p) - p(x)|)dx + M + \Phi(6|\varphi(0)|)|\Omega|. \tag{3.5.7}$$

On the other hand, if $\Phi(s) \geq 0$, one has, with (3.5.6b) and the first inequality of (3.5.1),

$$\frac{s}{3} = \varphi(\Phi(s)) \geq a(\Phi(s))^\gamma - b,$$

and then $\Phi(s) \leq (\frac{|s|}{3a} + \frac{b}{a})^{\frac{1}{\gamma}} \leq (\frac{|s|}{3a} + \frac{b}{a} + 1)^2$. This inequality gives an estimate on $\int_\Omega \Phi(3|m(p) - p(x)|)dx$ from the L^2-estimate on $(p - m(p))$. We hence get, with (3.5.7), an estimate on $\Phi(m(p))$. Using (3.5.6c) yields an estimate on $m(p)$. Finally, the estimate on $[m(p)]$ and $[p - m(p)]$ gives the existence of C_2 (depending on the data and θ_0) such that

$$\|p\|_{\mathrm{L}^2(\Omega)} \leq C_2. \tag{3.5.8}$$

Finally, thanks to $p = \varphi(\rho)$ and the first inequality of (3.5.1), the estimate on ρ follows. For the estimate on $|\rho|_\mathcal{T}$, which comes form the T_K term in (3.5.2b), the proof is similar to that given in lemma 3.4.3.

Passing to the limit in the discrete problem

Theorem 3.5.3 *Let a sequence of triangulations $(\mathcal{T}^{(n)})_{n\in\mathbb{N}}$ of Ω be given. We assume that h_n tends to zero when $n \to \infty$. In addition, we assume that the sequence of discretizations is regular, in the sense that there exists $\theta_0 > 0$ such that $\theta_n \geq \theta_0$, $\forall n \in \mathbb{N}$. For $n \in \mathbb{N}$, we denote by $\boldsymbol{W}_h^{(n)}$ and $L_h^{(n)}$ respectively the discrete spaces for the velocity and the pressure associated to $\mathcal{T}^{(n)}$ and by $(\boldsymbol{u}_n, p_n, \rho_n) \in \boldsymbol{W}_h^{(n)} \times L_h^{(n)} \times L_h^{(n)}$ a corresponding solution to the discrete problem (3.3.5), with $\alpha \geq 1$ and $0 < \xi < 2$. Then, up to the extraction of a subsequence, when $n \to \infty$:*

1. *up to the extraction of a subsequence, the sequence* $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ *(strongly) converges in* $\mathrm{L}^2(\Omega)^d$ *to a limit* $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega)^d$ *when* $n \to \infty$, $(p_n)_{n\in\mathbb{N}}$ *converges to* p *weakly in* $\mathrm{L}^2(\Omega)$ *and* ρ_n *converges to* ρ *weakly in* $\mathrm{L}^{2\gamma}(\Omega)$ *;*
2. $(\boldsymbol{u}, p, \rho)$ *are solution to Problem* (3.2.4a)–(3.2.4c).

Proof The first item of Theorem 3.5.3, namely the convergence (up to a subsequence) of the sequence $(\boldsymbol{u}_n, p_n, \rho_n)$ is a consequence of the uniform estimates of Theorem 3.5.2. The fact that $\rho \geq 0$ a.e. in Ω, $\int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M$ and $(\boldsymbol{u}, p, \rho)$ are solution of the momentum and mass-balance equation, the proof is similar to the same result in theorem 3.4.4.

Then, we only need here to prove that the equation of state is satisfied, that is $p = \varphi(\rho)$ a.e. in Ω.

The fact that $\rho \in \mathrm{L}^{2\gamma}(\Omega)$, $\rho \geq 0$ a.e. in Ω, $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega)^d$ and that $(\rho, \boldsymbol{u})$ satisfies (3.2.4b) yields, using Lemma 3.7.1 :

$$\int_\Omega \rho \,\mathrm{div}(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = 0. \tag{3.5.9}$$

Then, using (3.5.9), we have, following the proof given in [6] :

$$\lim_{n\to\infty} \int_\Omega \big(p_n - \mathrm{div}_h(\boldsymbol{u}_n)\big)\, \rho_n \,\mathrm{d}\boldsymbol{x} - \int_\Omega p\,\rho \,\mathrm{d}\boldsymbol{x} = 0.$$

As in [6], we also have $\limsup_{n\to\infty} \int_\Omega \mathrm{div}_h(\boldsymbol{u}_n)\, \rho_n \,\mathrm{d}\boldsymbol{x} \leq 0$. Hence :

$$\limsup_{n\to\infty} \int_\Omega p_n\, \rho_n \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega p\,\rho \,\mathrm{d}\boldsymbol{x}. \tag{3.5.10}$$

We want to deduce from (3.5.10) that $p = \varphi(\rho)$. But, since φ in only nondecreasing (and not necessarily increasing), we cannot use the proof given in [6]. We use here the so called Minty trick.

For simplicity, we define φ on $\mathbb{R}^-$ setting $\varphi(s) = \varphi(0)$ if $s < 0$. Let $\bar\rho \in L^{2\gamma}$ and, for $n \in \mathbb{N}$, $G_n = (\varphi(\rho_n) - \varphi(\bar\rho))(\rho_n - \bar\rho)$. One has $G_n \geq 0$ a.e. in Ω (since φ is nondecreasing). Thanks to the second inequality of (3.2.1) (which is used only in this proof) one has $\varphi(\bar\rho) \in L^{2\gamma/(2\gamma-1)}(\Omega)$ and then $\varphi(\bar\rho)\bar\rho \in L^1(\Omega)$. Then, one has $G_n \in L^1(\Omega)$ and

$$0 \leq \int_\Omega G_n \,dx = \int_\Omega (p_n\rho_n - p_n\bar\rho - \varphi(\bar\rho)\rho_n + \varphi(\bar\rho)\bar\rho) \,\mathrm{d}\boldsymbol{x}.$$

Using (3.5.10) and the weak convergences of p_n to p and ρ_n to ρ in $L^2(\Omega)$ and $L^{2\gamma}(\Omega)$ respectively, we obtain :

$$0 \leq \limsup_{n\to\infty} \int_\Omega G_n \,dx \leq \int_\Omega (p - \varphi(\bar\rho))(\rho - \bar\rho) \,\mathrm{d}\boldsymbol{x}.$$

We have thus proven that

$$\int_\Omega (p - \varphi(\bar\rho))(\rho - \bar\rho) \,\mathrm{d}\boldsymbol{x} \geq 0 \text{ for all } \bar\rho \in L^{2\gamma}(\Omega). \tag{3.5.11}$$

We now have to choose $\bar{\rho}$ conveniently to deduce $p = \varphi(\rho)$ from (3.5.11). Let $\psi \in C_c^\infty(\Omega, \mathbb{R})$. For $n \in \mathbb{N}^\star$, we set $\rho_n = \rho + \frac{1}{n}\psi$. Since $\rho_n \in L^{2\gamma}$, we can choose $\bar{\rho} = \rho_n$ in (3.5.11). We obtain

$$\int_\Omega (p - \varphi(\rho + \frac{1}{n}\psi))\psi \leq 0.$$

We now use the Dominated Convergence Theorem on the sequence $(g_n)_{n\in\mathbb{N}^\star}$ with $g_n = (p - \varphi(\rho + \frac{1}{n}\psi))\psi$. The continuity of φ gives
$g_n \to (p - \varphi(\rho))\psi$ a.e. in Ω. Since φ is nondecreasing, one has, for all $n \in \mathbb{N}^\star$,

$$|g_n| \leq G = |p\psi| + |\varphi(\rho + \|\psi\|_\infty)\psi| + |\varphi(0)\psi| \text{ a.e. in } \Omega.$$

The second inequality of (3.2.1) gives $\varphi(\rho + \|\psi\|_\infty) \in L^1(\Omega)$. Then one has $G \in L^1(\Omega)$ and the Dominated Convergence Theorem yields $\int_\Omega (p - \varphi(\rho))\psi \leq 0$. Changing ψ in $-\psi$, we conclude that $\int_\Omega (p - \varphi(\rho))\psi = 0$ for all $\psi \in C_c^\infty(\Omega, \mathbb{R})$. This gives $p = \varphi(\rho)$ a.e. in Ω, which concludes the proof.

Conclusion

We gave a scheme for the discretization of the stationary compressible Stokes problem with a general EOS and we proved the existence of a solution of the scheme along with the convergence of the approximate solution to an exact solution (up to a subsequence) as the mesh size goes to zero. A first difficulty of the paper is to get some estimates on the approximate solution (in particular with the dependancy of the forcing term with the density). A second complication is in the passage to the limit in the EOS. This difficulty is due to the nonlinearity of the EOS and the fact that the estimates on pressure and density only lead to weak convergences.

Appendix

3.6 Estimate on p

Lemma 3.6.1 *Let Ω be a bounded set of $\mathbb{R}^d$ $(d \geq 1)$ and $p \in L^2(\Omega)$, $p \geq 0$ a.e.. We assume that there exist $a < 1$ and $b \in \mathbb{R}$ such that*

$$\|p - m\|_{L^2} \leq a\|p\|_{L^2} + b,$$

where m is the mean value of p. Furthermore, we assume that there exist $A \in \mathbb{R}$ and a continuous function ψ from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\int_\Omega \psi(p)dx \leq A$ and $\lim_{s\to\infty} \psi(s) = +\infty$. Then, there exists C only depending on Ω, a, b A and ψ such that $\|p\|_{L^2} \leq C$.

Proof

We first modify the function ψ. Let $s_0 \in \mathbb{R}_+$ such that $\psi(s_0) > 0$. We define $\bar{\psi}$ by

$$\bar{\psi}(s) = \psi(s_0) \text{ if } 0 \leq s \leq s_0,$$
$$\bar{\psi}(s) = s \min_{t\in[s_0,s]} \frac{\psi(t)}{t} \text{ if } s_0 < s.$$

We remark that $\bar{\psi}(s) \leq \psi(s)$ for $s \geq s_0$, so that

$$\int_\Omega \bar{\psi}(p)dx \leq \bar{A} = A + \psi(s_0)|\Omega|.$$

Furthermore, one has $\lim_{s\to+\infty} \bar{\psi}(s) = +\infty$. In order to prove this result, let $(s_n)_{n\in\mathbb{N}}$ be an incresing sequence such that $\lim_{n\to\infty} s_n = +\infty$. For $n \in \mathbb{N}$ let $t_n \in [s_0, s_n]$ such that $\bar{\psi}(s_n) = (\psi(t_n)/t_n)s_n$. For any converging (in $\mathbb{R}_+ \cup \{+\infty\}$) subsequence of the sequence $(t_n)_{n\in\mathbb{N}}$, still denoted $(t_n)_{n\in\mathbb{N}}$, we have two possible cases,

First case. $\lim_{n\to\infty} t_n = t \in \mathbb{R}_+$. Then $\lim_{n\to\infty} \bar{\psi}(s_n) = +\infty$ (since $\psi(t)/t > 0$)

Second case. $\lim_{n\to\infty} t_n = +\infty$. Then $\lim_{n\to\infty} \bar{\psi}(s_n) = +\infty$ since $\bar{\psi}(s_n) \geq \psi(t_n)$.

We then conclude that $\lim_{s\to+\infty} \bar{\psi}(s) = +\infty$. Finally we also remark that the function $s \mapsto \frac{\bar{\psi}(s)}{s}$ is nonincreasing on $\mathbb{R}_+$.

We now prove the bound on $\|p\|_{L^2}$. Let $N > 0$, one has

$$\int_\Omega p(x)dx = \int_{p\geq N} p(x)dx + \int_{p<N} p(x)dx \leq \frac{1}{N}\int_\Omega p^2(x)dx + \frac{N}{\bar{\psi}(N)}\int_\Omega \bar{\psi}(p(x))dx.$$

This gives $m|\Omega| \leq \frac{1}{N}\|p\|_{L^2}{}^2 + \frac{N}{\bar{\psi}(N)}\bar{A}$. We now use the bound on $\|p-m\|_{L^2}$, it leads to

$$\begin{aligned}\|p\|_{L^2} &\leq \|p-m\|_{L^2} + m|\Omega|^{1/2} \\ &\leq a\|p\|_{L^2} + b + \frac{1}{N|\Omega|^{1/2}}\|p\|_{L^2}{}^2 + \frac{N}{\bar{\psi}(N)|\Omega|^{1/2}}\bar{A}.\end{aligned}$$

If $\|p\|_{L^2} \neq 0$, we now choose N such that $\frac{1}{N|\Omega|^{1/2}} = \frac{1-a}{2\|p\|_{L^2}}$, that is $N = \frac{2\|p\|_{L^2}}{(1-a)|\Omega|^{1/2}}$, we obtain

$$\frac{1-a}{2}\|p\|_{L^2} \leq b + \frac{2\bar{A}}{\bar{\psi}(N)(1-a)|\Omega|}\|p\|_{L^2}.$$

Since $\lim_{s\to\infty} \bar{\psi}(s) = +\infty$, there exists C_1 such that

$$N \geq C_1 \Rightarrow \frac{2\bar{A}}{\bar{\psi}(N)(1-a)|\Omega|} \leq \frac{1-a}{4}.$$

Then, with C_2 such that $\frac{2C_2}{(1-a)|\Omega|^{1/2}} = C_1$, ona has

$$\|p\|_{L^2} \geq C_2 \Rightarrow \frac{2\bar{A}}{\bar{\psi}(N)(1-a)|\Omega|} \leq \frac{1-a}{4}.$$

Therefore

$$\|p\|_{L^2} \geq C_2 \Rightarrow \|p\|_{L^2} \leq \frac{4b}{1-a}.$$

Then, we conclude that $\|p\|_{L^2} \leq C = \max\{C_2, \frac{4b}{1-a}\}$.

3.7 Passing to the limit in the EOS

Lemma 3.7.1 *Let Ω be a bounded open set of $\mathbb{R}^d$. Let $\rho \in L^2(\Omega)$, $\rho \geq 0$ a.e. in Ω and $u \in H_0^1(\Omega)^d$. Assume that (ρ, u) satisfies :*

$$\int_\Omega \rho\, \boldsymbol{u} \cdot \boldsymbol{\nabla}\varphi dx = 0 \textit{ for all } \varphi \in W^{1,\infty}(\Omega). \tag{3.7.1}$$

Then,

$$\int_\Omega \rho \operatorname{div}(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} = 0. \tag{3.7.2}$$

Remark 3.7.2 *Before giving the proof of Lemma 3.7.1, we want to point out the following remark. In the case of a regular function ρ, say $\rho \in C^1(\bar{\Omega})$, and assuming that $\rho > 0$ in Ω, the proof is very easy. We take $\varphi = \ln(\rho)$ in (3.7.1) which yields, since $\boldsymbol{\nabla}\varphi = \frac{1}{\rho}\boldsymbol{\nabla}\rho$,*

$$\int_\Omega u \cdot \boldsymbol{\nabla}\rho \, \mathrm{d}\boldsymbol{x} = 0.$$

But, for any $v \in C_c^\infty(\Omega)^d$ one has $\int_\Omega v \cdot \boldsymbol{\nabla}\rho dx = -\int_\Omega \rho \operatorname{div}(v) \, \mathrm{d}\boldsymbol{x}$. Then, the density of $C_c^\infty(\Omega)^d$ in $H_0^1(\Omega)^d$ yields $\int_\Omega v \cdot \boldsymbol{\nabla}\rho dx = -\int_\Omega \rho \operatorname{div}(v) \, \mathrm{d}\boldsymbol{x}$ for $v \in H_0^1(\Omega)^d$. This gives (3.7.2).

This proof is interesting because it suggests the proof of an equivalent result in the case of a discrete version (using a convenient numerical scheme) of $\operatorname{div}(\rho u) = 0$ (see Lemma 3.4.1). In other words, working on a numerical scheme is quite similar of working on the continuous equation with a regular solution.

Proof We now prove Lemma 3.7.1. (without assuming $\rho \in C^1(\bar{\Omega})$ and $\rho > 0$).

We set $u = 0$ in $\mathbb{R}^d \setminus \Omega$ and $\rho = 0$ in $\mathbb{R}^d \setminus \Omega$, we have $\rho \in L^2(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)^d$. We also deduce from (3.7.1) :

$$\int_{\mathbb{R}^d} \rho u \cdot \boldsymbol{\nabla}\varphi dx = 0 \text{ for all } \varphi \in C^1(\mathbb{R}^d). \tag{3.7.3}$$

Let $(r_n)_{n \in \mathbb{N}^\star}$ be a sequence of mollifiers, that is :

$$\begin{aligned} r \in C_c^\infty(\mathbb{R}^d, \mathbb{R}), \int_{\mathbb{R}^d} r dx = 1, \ r \geq 0 \text{ in } \mathbb{R}^d \\ \text{and, for } n \in \mathbb{N}^\star,\ x \in \mathbb{R}^d,\ r_n(x) = n^d r(nx). \end{aligned} \tag{3.7.4}$$

For $n \in \mathbb{N}^\star$, we set $\rho_n = \rho \star r_n$ and $(\rho u)_n = (\rho u) \star r_n$. Thanks to (3.7.3), we have $\operatorname{div}((\rho u)_n) = 0$ in $\mathbb{R}^d$. Since $u \in H^1(\mathbb{R}^d)^d$ and $\rho \in L^2(\mathbb{R}^d)$, we will prove in Lemma 3.7.4 that $\operatorname{div}((\rho u)_n - \rho_n u) \to 0$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Then, if $(q_n)_{n \in \mathbb{N}^\star}$ is a bounded sequence in $L^\infty(\mathbb{R}^d)$ which converges a.e. to q, we have :

$$-\int_{\mathbb{R}^d} \operatorname{div}(\rho_n u) \, q_n dx = \int_{\mathbb{R}^d} \operatorname{div}((\rho u)_n - \rho_n u) \, q_n dx \to 0 \text{ as } n \to \infty. \tag{3.7.5}$$

Let ψ be a bounded and C^1 function from $\mathbb{R}$ to $\mathbb{R}$, taking $q_n = \psi(\rho_n)$ in (3.7.5) (which converges a.e. to $\psi(\rho)$, at least up to a subsequence) we obtain

$$-\int_{\mathbb{R}^d} \operatorname{div}(\rho_n u)\, \psi(\rho_n) dx \to 0 \text{ as } n \to \infty.$$

We now define θ by $\theta(s) = \int_0^s t\psi'(t)\,\mathrm{d}\boldsymbol{t}$ for $s \in \mathbb{R}$ and we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \theta(\rho_n)\operatorname{div}(u) dx = \int_{\mathbb{R}^d} \rho_n \psi'(\rho_n) u \cdot \boldsymbol{\nabla}\rho_n dx &= \int_{\mathbb{R}^d} \rho_n u \cdot \boldsymbol{\nabla}\psi(\rho_n) dx \\ &= -\int_{\mathbb{R}^d} \operatorname{div}(\rho_n u)\, \psi(\rho_n)\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

and then $\displaystyle\int_{\mathbb{R}^d} \theta(\rho)\operatorname{div}(u) dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \theta(\rho_n)\operatorname{div}(u)\,\mathrm{d}\boldsymbol{x} = 0.$

It is now quite easy to construct a sequence of functions $(\psi_n)_{n\in\mathbb{N}}$ such that $0 \le \theta_n(s) \le s$ for all $s \in \mathbb{R}_+$ and $\lim_{n\to\infty} \theta_n(s) = s$ for all $s \in \mathbb{R}_+$. With the Dominated Convergence Theorem we then conclude that $\displaystyle\int_{\mathbb{R}^d} \rho \operatorname{div}(u)\,\mathrm{d}\boldsymbol{x} = 0.$

Remark 3.7.3 *Under the hypothesis of Lemma 3.7.1, a quick look on the proof of this lemma shows that it is also possible to prove*

$$\int_\Omega \psi(\rho)\operatorname{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = 0,$$

for any continuous function ψ (from $\mathbb{R}$ to $\mathbb{R}$) "at most linear", that is such that

$$\limsup_{s\to+\infty} \frac{|\psi(s)|}{s} < +\infty.$$

It is also possible (as it was said in Remark 3.2.3) to prove that (ρ, u) is a renormalized solution to $\operatorname{div}(\rho u) = 0$ in $\mathbb{R}^d$.

Indeed, let ψ be a bounded and C^1 function from $\mathbb{R}$ to $\mathbb{R}$ and $\varphi \in C_c^\infty(\Omega)$. Taking $q_n = \psi(\rho_n)\varphi$ in (3.7.5) (which converges a.e. to $\psi(\rho)\varphi$, at least up to a subsequence) we obtain

$$-\int_{\mathbb{R}^d} \operatorname{div}(\rho_n u)\, \psi(\rho_n)\varphi\,\mathrm{d}\boldsymbol{x} \to 0 \textit{ as } n \to \infty.$$

Taking, for $s \in \mathbb{R}$, $\bar\psi(s) = \int_0^s \psi(t)\,\mathrm{d}\boldsymbol{t}$ and $\theta(s) = \int_0^s t\psi'(t)\,\mathrm{d}\boldsymbol{t} = s\psi(s) - \bar\psi(s)$, we obtain, after some integrations by parts and passing to the limit as $n \to \infty$,

$$\int_{\mathbb{R}^d} (\rho\bar\psi'(\rho) - \bar\psi(\rho))(\operatorname{div} u)\varphi\,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^d} \bar\psi(\rho) u \cdot \boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} = 0.$$

Then, it is easy to see that this equality also holds if $\bar\psi$ is a C^1 fonction form $\mathbb{R}$ to $\mathbb{R}$ with a bounded derivative. This proves that (ρ, u) is a renormalized solution to $\operatorname{div}(\rho u) = 0$ in $\mathbb{R}^d$.

Lemma 3.7.4 *Let $\rho \in L^2(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)^d$. Let $(r_n)_{n\in\mathbb{N}^\star}$ be a sequence of mollifiers as given by* (3.7.4) *and, for $n \in \mathbb{N}^\star$, $\rho_n = \rho \star r_n$ and $(\rho u)_n = (\rho u) \star r_n$. Then,*

$$\boldsymbol{\nabla}((\rho u)_n - \rho_n u) \to 0 \textit{ in } L^1(\mathbb{R}^d)^{d\times d},$$

and then,

$$\operatorname{div}((\rho u)_n - \rho_n u) \to 0 \textit{ in } L^1(\mathbb{R}^d).$$

Proof

Let $i, j \in \{1, \dots, d\}$. Denoting by $u_1, \dots, u_d$ the components of u and by ∂_i the derivative with respect to x_i, we have to prove that the sequence $(\partial_i[(\rho u_j)_n - \rho_n u_j])_{n\in\mathbb{N}^\star}$ converges to 0 in $L^1(\mathbb{R}^d)$. (As a consequence, taking $i = j$ and summing on i, we obtain that $\operatorname{div}((\rho u)_n - \rho_n u) \to 0$ in $L^1(\mathbb{R}^d)$.)

We have

$$\partial_i[(\rho u_j)_n - \rho_n u_j] = (\rho u_j) \star \partial_i r_n - (\rho \star \partial_i r_n)\ u_j - \rho_n \partial_i u_j = F_n - G_n,$$

with

$$F_n = (\rho u_j) \star \partial_i r_n - (\rho \star \partial_i r_n)\ u_j - \rho(\partial_i u_j \star r_n)$$

and

$$G_n = \rho_n \partial_i u_j - \rho(\partial_i u_j \star r_n).$$

Since $\rho_n \to \rho$ in $L^2(\mathbb{R}^d)$ and $\partial_i u_j \star r_n \to \partial_i u_j$ in $L^2(\mathbb{R}^d)$ (as $n \to \infty$), the two parts of G_n converges in $L^1(\mathbb{R}^d)$ (as $n \to \infty$) to $\rho \partial_i u_j$. Then, the sequence $(G_n)_n$ converges in $L^1(\mathbb{R}^d)$ (as $n \to \infty$) to 0. It remains to show that $F_n \to 0$ in $L^1(\mathbb{R}^d)$.

Using the fact that $\partial_i u_j \star r_n = u_j \star \partial_i r_n$ and the fact that r_n has a compact support, we have, for a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned} F_n(x) &= \int_{\mathbb{R}^d} (\rho(x-y) - \rho(x))\ \big(u_j(x-y) - u_j(x)\big)\ \partial_i r_n(y) dy \\ &= \int_B (\rho(x - \frac{z}{n}) - \rho(x))\ \big(u_j(x - \frac{z}{n}) - u_j(x)\big)\ n\ \partial_i r(z) dz, \end{aligned}$$

where B is a ball of center 0 and radius R containing the support of r. Then, we get :

$$|F_n(x)| \le n \int_B |(\rho(x - \frac{z}{n}) - \rho(x))\ \big(u_j(x - \frac{z}{n}) - u_j(x)\big)|\ |\partial_i r(z)| dz.$$

We integrate over $\mathbb{R}$ the preceding inequality and we use the Fubini-Tonelli Theorem,

$$\begin{aligned} &\int_{\mathbb{R}^d} |F_n(x)| dx \le \\ &n \int_B \left[\int_{\mathbb{R}^d} |(\rho(x - \frac{z}{n}) - \rho(x))(u_j(x - \frac{z}{n}) - u_j(x))| dx \right] |\partial_i r(z)| dz. \end{aligned} \tag{3.7.6}$$

Using the Cauchy-Schwarz Inequality, we have for $z \in B$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |(\rho(x-\frac{z}{n})-\rho(x))(u_j(x-\frac{z}{n})-u_j(x))|dx \\
&\leq \left[\int_{\mathbb{R}^d} |\rho(x-\frac{z}{n})-\rho(x)|^2 dx\right]^{1/2} \left[\int_{\mathbb{R}^d} |u_j(x-\frac{z}{n})-u_j(x)|^2 dx\right]^{1/2}.
\end{aligned}
$$

For all $z \in B$ (see Lemma 3.7.5) we have

$$
\int_{\mathbb{R}^d} |u_j(x-\frac{z}{n})-u_j(x)|^2 dx \leq (\frac{R}{n})^2 \|u\|^2_{H^1(\mathbb{R}^d)^d}.
$$

Then,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |(\rho(x-\frac{z}{n})-\rho(x))(u_j(x-\frac{z}{n})-u_j(x))|dx \\
&\leq \frac{R}{n}\, \|u\|_{H^1(\mathbb{R}^d)^d} \left[\int_{\mathbb{R}^d} |\rho(x-\frac{z}{n})-\rho(x)|^2 dx\right]^{1/2}.
\end{aligned}
\tag{3.7.7}
$$

Let $\varepsilon > 0$. Since $\rho \in L^2(\mathbb{R}^d)$, there exists $\delta > 0$ such that

$$
h \in \mathbb{R}^d,\ |h| \leq \delta \Rightarrow \ \|\rho(\cdot + h) - \rho\|_{L^2(\mathbb{R}^d)} \ \leq \varepsilon.
$$

With (3.7.7), this gives if $n \geq R/\delta$ and $z \in B$,

$$
\int_{\mathbb{R}^d} |(\rho(x-\frac{z}{n})-\rho(x))(u_j(x-\frac{z}{n})-u_j(x))|dx \leq \varepsilon\frac{R}{n}\, \|u\|_{H^1(\mathbb{R}^d)^d}.
$$

Coming back to (3.7.6), we obtain, if $n \geq R/\delta$,

$$
\int_{\mathbb{R}^d} |F_n(x)|dx \leq n\frac{R}{n}\varepsilon\, \|u\|_{H^1(\mathbb{R}^d)^d} \int_B |\partial_i r(z)|dz = \varepsilon\, R\, \|u\|_{H^1(\mathbb{R}^d)^d} \int_B |\partial_i r(z)|dz.
$$

This proves that $F_n \to 0$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$ and concludes the proof of Lemma 3.7.4.

Lemma 3.7.5 *Let $w \in H^1(\mathbb{R}^d)$. Then, for $h \in \mathbb{R}^d$,*

$$
\|w(\cdot + h) - w\|_{L^2(\mathbb{R}^d)} \ \leq |h|\, \|w\|_{H^1(\mathbb{R}^d)}\,,
\tag{3.7.8}
$$

where $|h|$ is the Euclidean norm of h.

Lemma 3.7.5 is well-known. A proof is given, for instance, in [6].

The following lemma (Lemma 3.7.6) proves that (for regular enough set Ω) in Lemma 3.7.1, $W^{1,\infty}(\Omega)$ can be replaced by $C_c^\infty(\Omega)$. That is to say that 3.7.1 is true with $\varphi \in W^{1,\infty}(\Omega)$ if (and only if) it is true with the weaker assumption $\varphi \in C_c^\infty(\Omega)$. Lemma 3.7.6 is given with $\rho \in L^2(\Omega)$ and $u \in (H_0^1(\Omega))^d$, which is the case needed for the present paper (and allows a nice proof using the Hardy inequality). Similar results are possible with different assumptions on u and ρ (for instance, $\rho \in L^\infty(\Omega)$ and $u \in W_0^{1,1}(\Omega)$). However, the fact that $\rho u \in L^1(\Omega)$ is obviously not sufficient to ensure that (3.7.1) is true with $\varphi \in W^{1,\infty}(\Omega)$ as long as it is true for $\varphi \in C_c^\infty(\Omega)$. In a following paper, dealing with the Navier-Stokes equations, we will give the same lemma with a weaker assumption on ρ (since $\rho \notin L^2(\Omega)$ in the case of the Navier-Stokes equations, when $d = 3$ and $\gamma < \frac{5}{3}$). In this case, the proof will use a different argument, slightly more complicated.

Lemma 3.7.6 *Let Ω be a bounded open set of $\mathbb{R}^d$, with a Lipschitz continuous boundary. Let $u \in (H_0^1(\Omega))^d$ and $\rho \in L^2(\Omega)$ such that, for all $\varphi \in C_c^\infty(\Omega)$,*

$$\int_\Omega \rho u \cdot \boldsymbol{\nabla}\varphi dx = 0. \tag{3.7.9}$$

Then (3.7.9) *holds for all $\varphi \in W^{1,\infty}(\Omega)$.*

The proof of this lemma is given in [7] (Lemma A.1).

Remark 3.7.7 The hypothesis $\rho \in L^2(\Omega)$ is sharp in Lemma 3.7.6, as we will see now. Let $d > 1$and $2d/(d+2) < q < 2$. We give here an example for which (3.7.9) holds for all $\varphi \in C_c^\infty(\Omega)$ but does not hold for some $\varphi \in W^{1,\infty}(\Omega)$. In this example, one has $\rho \in L^q(\Omega)$ and $u \in (H_0^1(\Omega))^d$ (so that $\rho u \in L^1(\mathbb{R})^d$). Let us assume that $\Omega =]0,2[\times]-1,1[^{d-1}$. Let $\alpha \in]\frac{1}{2}, \frac{1}{q}[$. We define ρ and $u = (u_1, \ldots, u_d)^t$ as follows :

$$u_1(x) = x_1^\alpha \prod_{i=2}^{d} (1 - |x_i|) \text{ if } x \in \Omega,\ x_1 \leq 1,$$

$$u_1(x) = (2 - x_1)^\alpha \prod_{i=2}^{d} (1 - |x_i|) \text{ if } x \in \Omega,\ x_1 > 1,$$

$$u_2 = \ldots = u_d = 0,$$

$$\rho(x) = \frac{1}{x_1^\alpha} \text{ if } x \in \Omega,\ x_1 \leq 1,$$

$$\rho(x) = \frac{1}{(2 - x_1)^\alpha} \text{ if } x \in \Omega,\ x_1 > 1.$$

We have $\rho \in L^q(\Omega)$ (thanks to $\alpha q < 1$) and $u \in (H_0^1(\Omega))^d$ (thanks to $2\alpha > 1$). Since ρu_1 does not depend on x_1, if is easy to see (integrating by parts) that (3.7.9) holds for all $\varphi \in C_c^\infty(\Omega)$. Taking now $\varphi \in C_c^\infty(\mathbb{R}^d)$ with, for instance $\varphi = 0$ outside $]-1,1[\times]-\frac{1}{2},\frac{1}{2}[^{d-1}$, one has

$$\int_\Omega \rho u \cdot \nabla\varphi dx = -\int_{]-\frac{1}{2},\frac{1}{2}[^{d-1}} \prod_{i=2}^{d} (1 - |x_i|)\varphi(0, y) dy,$$

where $y = (x_2, \ldots, x_n)$. It is possible to choose φ such that $\varphi(0, y) > 0$ for all $y \in]-\frac{1}{2},\frac{1}{2}[^{d-1}$. This gives $\int_0 \rho u \cdot \nabla\varphi dx < 0$ and proves that (3.7.1) does not hold for this choice of φ (which belongs to $W^{1,\infty}(\Omega)$).

Lemma 3.7.8 *Let φ be a convex function from $\mathbb{R}_+$ to $\mathbb{R}_+$ and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative functions of $L^2(\Omega)$ weakly converging in $L^2(\Omega)$ to ρ. We assume that the sequence $(\varphi(\rho_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. Then, $\varphi(\rho) \in L^2(\Omega)$.*

Proof Since $\rho_n \geq 0$ a.e. (for all $n \in \mathbb{N}$), one has also $\rho \geq 0$ a.e..

Since the sequence $(\rho_n)_{n\in\mathbb{N}}$ weakly converge in $L^2(\Omega)$ to ρ, there exists a sequence $(\tilde{\rho}_n)_{n\in\mathbb{N}}$ converging (strongly) in $L^2(\Omega)$ to ρ and such that $\tilde{\rho}_n$ is (for all $n \in \mathbb{N}$) a convex combination of $\{\rho_k,\, k \geq n\}$ (this result is known as the Mazur lemma). Then, for all $n \in \mathbb{N}$, there exists $q_n \in \mathbb{N}$ and $\alpha_{n,0}, \ldots, \alpha_{n,q_n}$ such that

$$\tilde{\rho}_n = \sum_{i=0}^{q_n} \alpha_{n,i}\rho_{n+i},\ \sum_{i=0}^{q_n} \alpha_{n,i} = 1 \text{ and } \alpha_{n,i} \geq 0 \text{ for } i = 0, \ldots, q_n.$$

Let $M = \sup\{\, \|\varphi(\rho_n)\|_{L^2(\Omega)} \,\}$. Using the convexity of φ (and the fact that φ take its values in $\mathbb{R}_+$) we have, for all $n \in \mathbb{N}$,

$$0 \leq \varphi(\tilde{\rho}_n) \leq \sum_{i=0}^{q_n} \alpha_{n,i}\varphi(\rho_{n+i}) \text{ a.e.,}$$

and then

$$\|\varphi(\tilde{\rho}_n)\|_{L^2(\Omega)} \leq \sum_{i=0}^{q_n} \alpha_{n,i} \|\varphi(\rho_{n+i})\|_{L^2(\Omega)} \leq M.$$

Up to a subsequence, one has $\tilde{\rho}_n \to \tilde{\rho}$ a.e. and then, using the continuity of the function φ, $\varphi^2(\tilde{\rho}_n) \to \varphi^2(\rho)$ a.e on Ω. Then, using Fatou Lemma, we thus get $\varphi(\rho) \in L^2(\Omega)$ (and $\|\varphi(\rho)\|_{L^2(\Omega)} \leq M$).

3.8 General lemmas

Lemma 3.8.1 *Let $(F_n)_{n\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ be an equi-integrable sequence, and F be a function of $\mathrm{L}^1(\Omega)$. We assume that :*

$$\lim_{n\to\infty} \int_\Omega F_n\varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega F\varphi \,\mathrm{d}\boldsymbol{x} \text{ for all } \varphi \in \mathrm{C}_c^\infty(\Omega). \tag{3.8.1}$$

Then :

$$\lim_{n\to\infty} \int_\Omega F_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega F \,\mathrm{d}\boldsymbol{x}.$$

Lemma 3.8.1 is well-known. A proof is given, for instance, in [6]. The following lemma is also well-known. A simple proof of this result is given in [1].

Lemma 3.8.2 *Let $q \in \mathrm{L}^2(\Omega)$ such that $\int_\Omega q \,\mathrm{d}\boldsymbol{x} = 0$. Then, there exists $\boldsymbol{w} \in \mathrm{H}_0^1(\Omega)^d$ such that $\mathrm{div}(\boldsymbol{w}) = q$ a.e. in Ω and $\|\boldsymbol{w}\|_{\mathrm{H}^1(\Omega)^d} \leq c_2 \|q\|_{\mathrm{L}^2(\Omega)}$ where c_2 only depends on Ω.*

We now give two simple lemmas related to the so-called "M-matrices". We recall that for a vector x of $\mathbb{R}^n$, the fact that all the components of x are nonnegative is denoted by $x \geq 0$. Similarly the fact that all the components of x are positive is denoted by $x > 0$.

Lemma 3.8.3 *Let $n \in \mathbb{N}^\star$ and A be a $n \times n$ matrix with real entries (these entries are denoted by $a_{i,j}$, i, $j = 1, \ldots, n$). We asume that A satisfies the following properties :*

$$\begin{cases} a_{i,j} \leq 0 \text{ for all } i, j \in \{1, \ldots, n\},\ i \neq j, \\ a_{i,i} + \sum_{j \neq i} a_{i,j} > 0 \text{ for all } i \in \{1, \ldots, n\}. \end{cases}$$

then,

$$x \in \mathbb{R}^n,\ A\,x \geq 0 \Rightarrow x \geq 0, \tag{3.8.2}$$

which is equivalent to say that A is invertible and that all the entries of A^{-1} are nonnegatives. Futhermore, one also has

$$x \in \mathbb{R}^n,\ A\,x > 0 \Rightarrow x > 0, \tag{3.8.3}$$

Proof The proof of (3.8.2) is very classical. We can do it, for instance, by contradiction. Let $x \in \mathbb{R}^n$ such that $Ax \geq 0$. We assume that $\alpha = \min\{x_i,\ i = 1, \ldots, n\} < 0$ (where the x_i are the components of x) and we choose $i_0 \in \{1, \ldots, n\}$ such that $x_{i_0} = \alpha$.

Since the i_0-component of Ax is nonnegative and since $x_{i_0} \leq x_i$ for all i, one has, thanks to the properties of A,

$$x_{i_0}(a_{i_0,i_0} + \sum_{j \neq i_0} a_{i_0,j}) \geq 0,$$

Which gives $x_{i_0} \geq 0$, in contradiction with $x_{i_0} = \alpha < 0$. This proves (3.8.2).

In order to prove (3.8.3). Let e be the vector of $\mathbb{R}^n$ whose all components are equal to 1. let $x \in \mathbb{R}^n$ such $Ax > 0$. Then, for $\varepsilon > 0$ small enough, one has $A(x - \varepsilon e) = Ax - \varepsilon Ae \geq 0$. Thanks to (3.8.2), one deduces $x - \varepsilon e \geq 0$ and this gives $x > 0$.

The second lemma is a little bit less classical but is a very simple consequence of the first one.

Lemma 3.8.4 *Let $n \in \mathbb{N}^\star$ and A be a $n \times n$ matrix with real entries (these entries are denoted by $a_{i,j}$, i, $j = 1, \ldots, n$). We asume that A satisfies the following properties :*

$$\begin{cases} a_{i,j} \leq 0 \text{ for all } i, j \in \{1, \ldots, n\},\ i \neq j, \\ a_{i,i} + \sum_{j \neq i} a_{j,i} > 0 \text{ for all } i \in \{1, \ldots, n\}. \end{cases}$$

then,

$$x \in \mathbb{R}^n,\ A\,x \geq 0 \Rightarrow x \geq 0 \tag{3.8.4}$$

and

$$x \in \mathbb{R}^n,\ A\,x > 0 \Rightarrow x > 0, \tag{3.8.5}$$

Proof The matrix A^t satisfies the properties of lemma 3.8.3. Then A^t is invertible and $(A^t)^{-1}$ has all its entries nonnegative. This gives that A is also invertible and has all its entries nonnegative since $(A^t)^{-1} = (A^{-1})^t$. This gives that A satisfies (3.8.4).

The proof of (3.8.5) is the same as the proof of (3.8.3) in lemma 3.8.3.

Lemma 3.8.5 *Let φ be a function of class C^1 from $\mathbb{R}_+^\star$ to $\mathbb{R}$. Let ψ from $\mathbb{R}_+^\star$ to $\mathbb{R}$ such that $s\psi'(s) = \varphi'(s)$ for all $s \in \mathbb{R}_+^\star$. Let $a, b \in \mathbb{R}_+^\star$, $a \neq b$. Then, there exists c between a et b such that*

$$(\psi(b) - \psi(a))b - (\varphi(b) - \varphi(a)) = \frac{1}{2}(b-a)^2\psi'(c).$$

Proof One has

$$(\psi(b) - \psi(a))b - (\varphi(b) - \varphi(a)) = b\int_a^b \psi'(s)ds - \int_a^b \varphi'(s)ds = \int_a^b (b-s)\psi'(s)ds.$$

But,

$$\min_{t\in[a,b]} \psi'(t) \int_a^b (b-s)ds \leq \int_a^b (b-s)\psi'(s)ds \leq \max_{t\in[a,b]} \psi'(t) \int_a^b (b-s)ds.$$

Then, since ψ' is continuous on $[a, b]$, there exists $c \in [a, b]$ such that

$$\int_a^b (b-s)\psi'(s)ds = \psi'(c) \int_a^b (b-s)ds.$$

Noticing that $\displaystyle\int_a^b (b-s)ds = \frac{1}{2}(b-a)^2$, we obtain the desired equality.

Chapitre 4

Approximation par viscosité des équations de Stokes compressibles

4.1 Abstract

In this chapter, we consider the continuous stationary compressible Stokes problem with a general equation of state of the form $p = \varphi(\rho)$ (where p stands for the pressure, ρ for the density and φ is a nondecreasing super-linear function belonging to $C^1(\mathbb{R}_+^\star, \mathbb{R})$). We prove the existence of a solution to this problem by passing to the limit on a regularized problem which the existence result will be previously demonstrated.

4.2 Introduction

Let Ω be a connected bounded open set of $\mathbb{R}^d$, ($d = 2$ or 3).

For $M > 0$, $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$, $\boldsymbol{g} \in L^\infty(\Omega)^d$ and $\varphi \in C(\mathbb{R}, \mathbb{R})$ a convex nondecreasing function such that :

$$\varphi(0) = 0, \ \varphi \text{ is } C^1 \text{ on } \mathbb{R}_+^\star$$

and

$$\forall a \in \mathbb{R}, \ \exists b > 0 \text{ such that : } \varphi(s) \geq as - b, \ \forall s \in \mathbb{R}_+. \tag{4.2.1}$$

We consider the following problem :

$$-\Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} + \rho \boldsymbol{g} \text{ in } \Omega, \quad \boldsymbol{u} = 0 \text{ on } \partial\Omega, \tag{4.2.2a}$$

$$\mathrm{div}(\rho \boldsymbol{u}) = 0 \ \text{ in } \Omega, \ \rho \geq 0 \ \text{ in } \Omega, \ \int_\Omega \rho(x) \,\mathrm{d}\boldsymbol{x} = M, \tag{4.2.2b}$$

$$p = \varphi(\rho) \text{ in } \Omega. \tag{4.2.2c}$$

Remark 4.2.1 – *the condition* (4.2.1) *is equivalent to the following one :*

$$\liminf_{s \to +\infty} \varphi(s)/s = +\infty$$

- *The fact that $\varphi(0) = 0$, is not a restriction since p can be replaced by $(p - \varphi(0))$ in the momemtum equation, and the EOS can be written as $p - \varphi(0) = \varphi(\rho) - \varphi(0)$.*
- *The convexity of the function φ and (4.2.1) can be replaced by the following condition : $\exists a, \tilde{a}, b, \tilde{b} > 0$ and $\gamma > 1$ such that :*

$$\forall s \in \mathbb{R}_+,\ as^\gamma - b \le \varphi(s) \le \tilde{a}s^{2\gamma-1} + \tilde{b}. \tag{4.2.3}$$

Definition 4.2.2 *Let $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$, $\boldsymbol{g} \in L^\infty(\Omega)^d$ and $M > 0$. A weak solution of Problem (4.2.2) is a function $(\boldsymbol{u}, p, \rho) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$ satisfying :*

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\mathrm{div}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x},\ \forall \boldsymbol{v} \in \mathrm{H}^1_0(\Omega)^d, \tag{4.2.4a}$$

$$\int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi \,\mathrm{d}\boldsymbol{x} = 0 \text{ for all } \psi \in W^{1,\infty}(\Omega), \tag{4.2.4b}$$

$$\rho \ge 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M,\ p = \varphi(\rho) \text{ a.e. in } \Omega. \tag{4.2.4c}$$

The objective of this paper is to prove that problem (4.2.4) admits a solution
by passing to the limit on the following regularized problem as m, k, n tend to $+\infty$,

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\mathrm{div}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega T_k(\rho) \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x},\ \forall \boldsymbol{v} \in \mathrm{H}^1_0(\Omega)^d, \tag{4.2.5a}$$

$$\int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi \,\mathrm{d}\boldsymbol{x} - \frac{1}{n} \int_\Omega \boldsymbol{\nabla} \rho(x) \cdot \boldsymbol{\nabla} \psi(x) dx = 0 \text{ for all } \psi \in W^{1,\infty}(\Omega), \tag{4.2.5b}$$

$$\rho \ge 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M,\ p = \varphi_m(\rho) \text{ a.e. in } \Omega. \tag{4.2.5c}$$

with $T_k(s) = \min(k, s)$ and $\varphi_m(s) = \min(m, \varphi(s))$ if $s > 0$
and $T_k(s) = \varphi_m(s) = 0$ if $s \le 0$.

To prove that (4.2.5) admits a solution, we first treat, in Section 2, the convection-diffusion problem where we prove the existence and uniqueness result with some useful properties.We then give in Section 3, the existence result of (4.2.5) and present the passage to the limit on this problem which consists the proof of the main result of this paper. Finally, in section A, we recall and prove some results that we use in the previous sections.

4.3 Convection-diffusion problem

We consider here the following equation

$$-\Delta \rho + \mathrm{div}(\rho u) = 0 \text{ in } \Omega, \tag{4.3.1}$$

with the natural boundary condition which reads, if Ω, u and ρ are regular enough, $-\nabla\rho\cdot n+\rho u\cdot n=0$ (where n is the exterior normal vector to the boundary of Ω).

Under the hypothesis $u\in L^p(\Omega)^d$ for some $p>d$, the weak formulation of this problem is

$$\begin{aligned}&\rho\in H^1(\Omega),\\&\int_\Omega \nabla\rho(x)\cdot\nabla\varphi(x)dx-\int_\Omega \rho(x)u(x)\cdot\nabla\varphi(x)dx=0 \text{ for all } \varphi\in H^1(\Omega).\end{aligned}\tag{4.3.2}$$

We give in Theorem 4.3.1 an existence and uniqueness result along with some useful properties.

Theorem 4.3.1 *Let Ω be a connected bounded open set of $\mathbb{R}^d$ ($d=2$ or 3) with a Lipschitz continuous boundary. Let $p>d$, $u\in L^p(\Omega)^d$ and $M\geq 0$. Then, there exist a unique weak solution to problem* (4.3.2) *satisfying the additional condition $\int_\Omega\rho(x)dx=M$. Furthermore one has the two following properties :*

1. *$\rho>0$ a.e. on Ω if $M>0$ and $\rho=0$ a.e. on Ω if $M=0$.*
2. *For any $A>0$, there exists C only depending on A, M, p, and Ω such that, if ρ is the solution of* (4.3.2) *with $\int_\Omega\rho(x)dx=M$, one has*

$$\|u\|_{L^p(\Omega)}\leq A\Rightarrow\|\rho\|_{H^1(\Omega)}\leq C.$$

Remark 4.3.2 It is possible to replace in (4.3.1) the right hand side (which is 0) by a function f in $L^2(\Omega)$ (or by T in the dual space of $H^1(\Omega)$) provided that $\int_\Omega f(x)dx=0$ (or $\langle T,1\rangle_{H^1(\Omega)',H^1(\Omega)}=0$). In this case, an existence and uniqueness result of a weak solution to (4.3.1) with the additional condition $\int_\Omega\rho(x)dx=M$ is also true.

Proof of Theorem 4.3.1

The proof is divided in 3 steps.

Step 1 proves a *a priori* positivity. Namely, if ρ satisfy (4.3.2) with $\int_\Omega\rho(x)dx=M$, then $\rho>0$ a.e. in Ω if $M>0$ and $\rho=0$ a.e. in Ω if $M=0$. This result gives, in particular, the uniqueness of the solution (but not the existence) of (4.3.2) with $\int_\Omega\rho(x)dx=M$.

Step 2 gives a *a priori* estimate on the solutions of (4.3.2) with $\int_\Omega\rho(x)dx=M$. Indeed, it gives the second property of Theorem 4.3.1 (but not yet the existence result).

Step 3 gives the desired existence result, using the Leray-Schauder topological degree. Actually, this existence result can also be viewed as a consequence of the Fredholm alternative.

Step 1, *a priori* **positivity and uniqueness**

Let ρ be a solution of (4.3.2) with $\int_\Omega\rho(x)dx=M$. In order to prove that $\rho>0$ a.e. if $M>0$, we argue by contradiction. We set $\omega=\{\rho\leq 0\}$ and we assume that $\lambda_d(\omega)>0$ (where λ_d is the Lebesgue measure on $\mathbb{R}^d$).

For $n\in\mathbb{N}^\star$ we define T_n from $\mathbb{R}$ to $\mathbb{R}$ by $T_n(s)=\min\{\frac{1}{n},\max\{s,0\}\}$. It is well known that the function $T_n(\rho)$ belongs to $H^1(\Omega)$ and that

$$\nabla T_n(\rho)=1_{0<\rho<\frac{1}{n}}\nabla\rho \text{ a.e. in } \Omega.$$

Then, taking $\varphi = T_n(\rho)$ in (4.3.2) leads to

$$\int_\Omega |\boldsymbol{\nabla} T_n(\rho)|^2 dx = \int_\Omega \rho u \cdot \boldsymbol{\nabla} T_n(\rho) dx \leq \frac{a_n}{n}\Big(\int_\Omega |\boldsymbol{\nabla} T_n(\rho)|^2 dx\Big)^{\frac{1}{2}},$$

with

$$a_n = \Big(\int_{0<\rho<\frac{1}{n}} |u|^2 dx\Big)^{\frac{1}{2}}.$$

Since $u \in L^2(\Omega)^d$ and $\lim_{n\to\infty} \lambda_d\{0 < \rho < \frac{1}{n}\} = 0$, one has $\lim_{n\to\infty} a_n = 0$. Using the fact that $\|v\|_{L^1(\Omega)} \leq \|v\|_{L^2(\Omega)} \lambda_d(\Omega)^{1/2}$, we have

$$\| |\boldsymbol{\nabla} T_n(\rho)| \|_{L^1(\Omega)} \leq \| |\boldsymbol{\nabla} T_n(\rho)| \|_{L^2(\Omega)} \lambda_d(\Omega)^{1/2} \leq \frac{a_n}{n} \lambda_d(\Omega)^{1/2}.$$

We now remark that $T_n(\rho) = 0$ a.e. on ω. Since $\lambda_d(\omega) > 0$, Lemma 4.4.12 (which uses the connexity of Ω)gives the existence of C, only depending on Ω and ω such that

$$\|T_n(\rho)\|_{L^1(\Omega)} \leq C \| |\boldsymbol{\nabla} T_n(\rho)| \|_{L^1(\Omega)} .$$

Since

$$\|T_n(\rho)\|_{L^1(\Omega)} \geq \frac{1}{n} \lambda_d(\{\rho \geq \frac{1}{n}\}),$$

we then have

$$\lambda_d(\{\rho \geq \frac{1}{n}\}) \leq C a_n \lambda_d(\Omega)^{1/2}.$$

Passing to the limit as $n \to \infty$ leads to $\lambda_d(\{\rho > 0\}) = 0$, that is $\rho \leq 0$ a.e..

If $M > 0$, it is impossible since $\int_\Omega \rho dx = M > 0$. Then, we conclude that that $\lambda_d(\omega) = 0$, which gives $\rho > 0$ a.e. in Ω.

If $M = 0$, one has $\int_\Omega \rho dx = M = 0$ and then from $\rho \leq 0$ a.e. we conlude that $\rho = 0$ a.e. in Ω.

It is now easy to prove the uniqueness of the solution of (4.3.2) with $\int_\Omega \rho dx = M$. Let ρ_1 and ρ_2 be two solutions of (4.3.2) with $\int_\Omega \rho_1 dx = \int_\Omega \rho_2 dx = M$. We set $\rho = \rho_1 - \rho_2$. Then, ρ is a solution of (4.3.2) with $\int_\Omega \rho dx = 0$. This gives $\rho = 0$ a.e., which is the desired uniqueness result.

Actually, it is interesting to notice that the present step consists to prove that any solution of (4.3.2) has a constant sign.

Step 2, ***a priori* estimate**

Let $A > 0$ and assume that $\|u\|_{L^p(\Omega)} \leq A$ and $0 \leq M \leq A$. Let ρ be a solution of (4.3.2) with $\int_\Omega \rho dx = M$.

Taking $\varphi = \rho$ in (4.3.2) and using Hölder Inequality with $q = 2p/(p-2)$ (which gives $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$) leads to

$$\| |\boldsymbol{\nabla} \rho| \|_{L^2(\Omega)}{}^2 = \int_\Omega |\boldsymbol{\nabla} \rho|^2 dx \leq \||u|\|_{L^p(\Omega)} \|\rho\|_{L^q(\Omega)} \| |\boldsymbol{\nabla} \rho| \|_{L^2(\Omega)} . \tag{4.3.3}$$

We Choose $\bar{q}$ such that $q < \bar{q} < +\infty$ if $d = 2$ and $\bar{q} = 6$ if $d = 3$ (which gives $q < \bar{q}$). By Sobolev Inequality, there exists $C_s > 0$ only depending on Ω such that

$$\|\rho\|_{L^{\bar{q}}(\Omega)} \leq C_s \|\rho\|_{H^1(\Omega)} .$$

By Hölder Inequality, we also have, with $\theta = \frac{\bar{q}-q}{q(\bar{q}-1)} \in (0,1)$ (which only depends on p and d),

$$\|\rho\|_{L^q(\Omega)} \leq \|\rho\|_{L^1(\Omega)}{}^{\theta} \, \|\rho\|_{L^{\bar{q}}(\Omega)}{}^{1-\theta}.$$

This gives, since $\|\rho\|_{L^1(\Omega)} = \int_\Omega \rho dx = M$,

$$\|\rho\|_{L^q(\Omega)} \leq M^\theta C_s^{1-\theta} \, \|\rho\|_{H^1(\Omega)}{}^{1-\theta},$$

and, with (4.3.3),

$$\| \, |\boldsymbol{\nabla}\rho| \, \|_{L^2(\Omega)} \leq A \, M^\theta C_s^{1-\theta} \, \|\rho\|_{H^1(\Omega)}{}^{1-\theta}. \tag{4.3.4}$$

We now use the Poincaré-Wirtinger Inequality. It gives the existence of $C_p > 0$ only depending on Ω such that

$$\|\rho - M\lambda_d(\Omega)^{-1}\|_{L^2(\Omega)} \leq C_p \, \| \, |\boldsymbol{\nabla}\rho \, |\|_{L^2(\Omega)}$$

(the connexity of Ω is used once again here). Then we have

$$\|\rho\|_{L^2(\Omega)} \leq \|\rho - M\lambda_d(\Omega)^{-1}\|_{L^2(\Omega)} + M\lambda_d(\Omega)^{-\frac{1}{2}} \leq C_p \, \| \, |\boldsymbol{\nabla}\rho| \, \|_{L^2(\Omega)} + A\lambda_d(\Omega)^{-\frac{1}{2}}.$$

This gives

$$\|\rho\|_{H^1(\Omega)} \leq 2(C_p + 1) \, \| \, |\boldsymbol{\nabla}\rho| \, \|_{L^2(\Omega)} + 2A\lambda_d(\Omega)^{-\frac{1}{2}}. \tag{4.3.5}$$

Finally, with (4.3.4) and (4.3.5), we obtain the existence of C_1 and C_2 only depending on A, M, p and Ω such that

$$\|\rho\|_{H^1(\Omega)} \leq C_1 \, \|\rho\|_{H^1(\Omega)}{}^{1-\theta} + C_2.$$

Since $\theta > 0$, this gives the existence of C only depending on A, M, p and Ω such that $\|\rho\|_{H^1(\Omega)} \leq C$ and concludes this step.

Step 3, existence

For u and M given, we have to prove the existence of a solution to (4.3.2) with $\int_\Omega \rho dx = M$.

Let $t \in [0,1]$ and $q = 2p/(p-2)$. We now define a continuous and compact application from $[0,1] \times L^q(\Omega)$ in $L^q(\Omega)$. For $t \in [0,1]$ and $\bar{\rho} \in L^q(\Omega)$, since $u\bar{\rho} \in L^2(\Omega)$, it is well known that there exists a unique weak solution of the following problem (which is the classical Neumann problem) :

$$\begin{aligned} &\rho \in H^1(\Omega), \ \int_\Omega \rho dx = 0, \\ &\int_\Omega \boldsymbol{\nabla}\rho \cdot \boldsymbol{\nabla}\varphi dx = \int_\Omega \bar{\rho} u \cdot \boldsymbol{\nabla}\varphi dx \text{ for all } \varphi \in H^1(\Omega). \end{aligned} \tag{4.3.6}$$

This solution continuously depends in $H^1(\Omega)$ on $\bar{\rho}$ in $L^q(\Omega)^d$.

We define

$$F(t, \bar{\rho}) = t(\rho + \frac{M}{\lambda_d(\Omega)})$$

so that $\int_\Omega F(t,\rho)dx = tM$. Then, if $\rho = F(1,\rho)$, the function ρ is a solution of (4.3.2) with $\int_\Omega \rho dx = M$. We prove below the existence of such a function ρ using the invariance by homotopy of the Leray-Schauder topological degree.

Thanks to the Sobolev Embedding Theorem, the space $H^1(\Omega)$ is continuously embedded in $L^q(\Omega)$ (since $q < 2d/(d-2)$). Then F is continuous from $[0,1] \times L^q(\Omega)$ to $L^q(\Omega)$.

Furthermore, since $H^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ (once again since $q < 2d/(d-2)$), the function F is compact from $[0,1] \times L^q(\Omega)$ to $L^q(\Omega)$.

Now, we remark that

$$(t \in [0,1],\ \rho \in L^q(\Omega),\ \rho = F(t,\rho)) \Rightarrow \rho \text{ is solution of problem (4.3.2) with } \int_\Omega \rho = t\,M$$

A quick look on step 2 gives an H^1 estimate on ρ, namley,

$$(t \in [0,1],\ \rho \in L^q(\Omega),\ \rho = F(t,\rho)) \Rightarrow \exists C > 0 \,\text{such that}\ \ \|\rho\|_{H^1(\Omega)} \le C.$$

Then, there exists $R > 0$ such that

$$(t \in [0,1],\ \rho \in L^q(\Omega),\ \rho = F(t,\rho)) \Rightarrow\ \|\rho\|_{L^q(\Omega)} < R.$$

Let B_R be the ball of radius R and center 0 in $L^q(\Omega)$. The topological degree of $Id - F(t,\cdot)$ (where Id is the application $\rho \mapsto \rho$) on B_R associated to point 0 is well defined and is independant of $t \in [0,1]$. This gives $d(Id - F(1,\cdot), B_R, 0) = d(Id - F(0,\cdot), B_R, 0)$. But, since $F(0,\cdot) = 0$, we have $d(Id - F(0,\cdot), B_R, 0) = 1$. Then $d(Id - F(1,\cdot), B_R, 0) = 1$. This proves the existence of $\rho \in B_R$ such that $\rho = F(1,\rho)$ and concludes the proof of the Theorem 4.3.1.

4.4 The regularized problem

4.4.1 Existence of a solution

We prove in this section the existence of a weak solution to problem(4.2.5).

We will apply the Brouwer Fixed Point Theorem to a convenient application $T : L^2(\Omega) \to L^2(\Omega)$.

Let $\tilde\rho \in L^2(\Omega)$ and p defined by $p = \varphi_m(\tilde\rho)$.

Knowing p and $\tilde\rho$, we have the existence of $\boldsymbol{u}$ solution of the following problem :

$$\begin{aligned}&\boldsymbol{u} \in \mathrm{H}_0^1(\Omega)^d,\\ &\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega p\ \mathrm{div}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega T_k(\tilde\rho)\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \text{ for all } \boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d.\end{aligned} \tag{4.4.1}$$

which the proof is classical by Lax-Milgram lemma.

Knowing $\boldsymbol{u}$, we now define ρ as the solution of the following problem :

$$\begin{aligned}&\rho \in H^1(\Omega),\\ &\int_\Omega \boldsymbol{\nabla}\rho(x)\cdot\boldsymbol{\nabla}\varphi(x)dx - \int_\Omega \rho(x)u(x)\cdot\boldsymbol{\nabla}\varphi(x)dx = 0 \text{ for all } \varphi \in H^1(\Omega).\end{aligned} \tag{4.4.2}$$

which the existence proof is given in section 4.3.
We now complete the definition of the application T as follows :

$$T : L^2(\Omega) \to L^2(\Omega)$$

$$\tilde{\rho} \to \rho$$

T satisfies the following properties :

$\bullet T$ is continuous :

Let $(\tilde{\rho_n})_{n\in\mathbb{N}}$ a sequence in $L^2(\Omega)$ and $\rho_n = T(\tilde{\rho_n})$.We suppose that $\tilde{\rho_n}$ converges to $\tilde{\rho}$ in $L^2(\Omega)$ and will prove that $\rho_n = T(\tilde{\rho}_n) \to T(\tilde{\rho})$ in $L^2(\Omega)$ as $n \to +\infty$.

We first have, up to a subsequence, $p_n = \varphi_m(\tilde{\rho_n}) \to \varphi_m(\tilde{\rho})$ a.e in Ω (since φ_m is continuous) and now, let $\boldsymbol{u}_n$ the solution of equation (4.4.1) with p_n and $\tilde{\rho}_n$ in the right hand side, we thus have $\exists\, C1$ only depending on m, k, $\boldsymbol{f}$ and $\boldsymbol{g}$ such that :

$$\|\boldsymbol{u}_n\|_{H^1_0(\Omega)^d} \leq C1 \tag{4.4.3}$$

and then, $\exists\, \boldsymbol{u} \in H^1_0(\Omega)^d$ such that (up to a subsequence)

$$\boldsymbol{u}_n \to \boldsymbol{u} \text{ in } L^2(\Omega) \text{ and weakly in } H^1_0(\Omega)^d.$$

Furthermore Sobolev embedding gives te existence of $C2$ only depending on Ω, m, k, $\boldsymbol{f}, \boldsymbol{g}$ and p for $p \in [1, +\infty)$ if $d = 2$ and $p = 6$ for $d = 3$, such that

$$\|\boldsymbol{u}_n\|_{L^p(\Omega)^d} \leq C2 \tag{4.4.4}$$

Let now ρ_n the solution of the following problem

$$\begin{aligned} &\rho_n \in H^1(\Omega), \int_\Omega \rho_n = M \\ &\int_\Omega \boldsymbol{\nabla}\rho_n(x)\cdot\boldsymbol{\nabla}\varphi(x)dx - \int_\Omega \rho_n(x)u_n(x)\cdot\boldsymbol{\nabla}\varphi(x)dx = 0 \text{ for all } \varphi \in H^1(\Omega). \end{aligned} \tag{4.4.5}$$

We thus get, using theorem 4.3.1 and inequality (4.4.4) $\exists\, C$ only depending on m, k, $\boldsymbol{f}$, $\boldsymbol{g}$ and M such that

$$\|\rho_n\|_{H^1(\Omega)} \leq C$$

and then, $\exists\, \rho \in H^1(\Omega)$ such that $\rho_n \to \rho$ in $L^2(\Omega)$ and weakly in $H^1(\Omega)$.

It remains now to prove that the limit ρ is the solution of (4.4.2). To do that, we pass to the limit on the following problem, as $n \to +\infty$,

$$\begin{aligned} &\int_\Omega \rho_n = M, \\ &\int_\Omega \boldsymbol{\nabla}\rho_n(x)\cdot\boldsymbol{\nabla}\varphi(x)dx - \int_\Omega \rho_n(x)\boldsymbol{u}_n(x)\cdot\boldsymbol{\nabla}\varphi(x)dx = 0 \text{ for all } \varphi \in W^{1,\infty}(\Omega). \end{aligned} \tag{4.4.6}$$

we thus get

$$
\begin{aligned}
&\int_\Omega \rho = M, \\
&\int_\Omega \boldsymbol{\nabla}\rho(x)\cdot\boldsymbol{\nabla}\varphi(x)dx - \int_\Omega \rho(x)\boldsymbol{u}(x)\cdot\boldsymbol{\nabla}\varphi(x)dx = 0 \text{ for all } \varphi\in W^{1,\infty}(\Omega).
\end{aligned}
\tag{4.4.7}
$$

We have thus proven that, up to a subsequence, $\rho_n = T(\tilde{\rho_n}) \to T(\tilde{\rho})$ in $L^2(\Omega)$.By a classical proof by contradiction, we can prove that this convergence is also true without extraction of a subsequence which concludes the proof.

$\bullet Im(T) \subset B_R = \{\rho \in H^1(\Omega) \text{ such that } \|\rho\|_{H^1} \leq R\}$

Indeed, taking $\boldsymbol{v} = \boldsymbol{u}$ in equation (4.4.1) yields :

$$\|\boldsymbol{u}\|_{H^1} \leq C(\Omega, \boldsymbol{f}, \boldsymbol{g}, m, k)$$

and then, by Sobolev embedding we get for some $p > d$, $\exists C(\Omega, \boldsymbol{f}, \boldsymbol{g}, m, k, p)$ such that

$$\|\boldsymbol{u}_n\|_{L^p(\Omega)^d} \leq C$$

and then using theorem 4.3.1, we get :

$$\exists R > 0 \text{ such that } \|\rho\|_{H^1} \leq R. \tag{4.4.8}$$

By Rellich Theorem and (4.4.8) we have that $Im(T)$ is relatively compact in $L^2(\Omega)$.
And then applying the Brouwer Fixed Point Theorem, we get :

$$\exists \rho \in B_R \text{ such that : } T(\rho) = \rho$$

Finally we thus obtain : $\exists(\boldsymbol{u}, p, \rho) \in H_0^1(\Omega)^d \times L^2(\Omega) \times H^1(\Omega)$ a solution of problem (4.2.5).

4.4.2 Passage to the limit on the regularized problem

• **Fixing** k, n **and tending** m **to** $+\infty$

We first prove the following result

Lemma 4.4.1 *Let* $n \in \mathbb{N}^\star, M > 0$, $\boldsymbol{u} \in \mathrm{H}_0^1(\Omega)^d$ *and* $\rho \in H^1(\Omega)$ *a solution of the following problem :*

$$
\begin{aligned}
&\int_\Omega \rho(x)u(x)\cdot\boldsymbol{\nabla}\psi(x)dx - \frac{1}{n}\int_\Omega \boldsymbol{\nabla}\rho(x)\cdot\boldsymbol{\nabla}\psi(x)dx = 0 \textit{ for all } \psi\in H^1(\Omega), \\
&\int_\Omega \rho(x)\,dx = M.
\end{aligned}
\tag{4.4.9}
$$

Then, $\rho > 0$ and for $\Phi \in C^0(\mathbb{R}^+, \mathbb{R})$ nondecreasing such that $\Phi(0) = 0$ and $\Phi(\rho) \in L^2(\Omega)$,

$$\int_\Omega \Phi(\rho)\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \leq 0 \tag{4.4.10}$$

Proof The positivity of the solution ρ of (4.4.9) results from theorem 4.3.1 .
We now give the proof of (4.4.10) which is composed of the following steps

Step 1 Let $\alpha, \eta, a \in \mathbb{R}_+^*$, we suppose in this case that Φ satisfies the following properties :

$$\begin{cases} \Phi \in C^1(\mathbb{R}), \text{nondecreasing} \\ \Phi = 0 \,\text{on}\,]-\infty, \eta] \\ \Phi = \alpha \,\text{on}\,]a, +\infty[\end{cases}$$

Let Ψ defined by $\Psi(s) = \displaystyle\int_0^s \frac{\Phi'(t)}{t}\,\mathrm{d}\boldsymbol{t}$.
Taking $\psi = \Psi \circ \rho \in H^1(\Omega)$ in equation (4.4.9)

$$\int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x} - \frac{1}{n}\int_\Omega \boldsymbol{\nabla}\rho(x) \cdot \boldsymbol{\nabla}\psi(x) dx = 0$$

we thus obtain :

$$\int_\Omega \boldsymbol{u} \cdot \Phi'(\rho)\boldsymbol{\nabla}\rho\,\mathrm{d}\boldsymbol{x} - \frac{1}{n}\int_\Omega |\boldsymbol{\nabla}\rho|^2 \frac{\Phi'(\rho)}{\rho}\,\mathrm{d}\boldsymbol{x} = 0$$

Using the fact that the function Φ is nondecreasing and $\rho > 0$, we get :

$$\int_\Omega \boldsymbol{u}\,\Phi'(\rho)\boldsymbol{\nabla}\rho\,\mathrm{d}\boldsymbol{x} \geq 0$$

and then since $\boldsymbol{u} \in H_0^1(\Omega)^d$,

$$\int_\Omega \Phi(\rho)\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \leq 0$$

Step 2 In this step we take Φ satisfying

$$\begin{cases} \Phi \in C^0(\mathbb{R}), \text{nondecreasing} \\ \Phi = 0 \,\text{on}\,]-\infty, \eta] \\ \Phi = \alpha \,\text{on}\, [a, +\infty[\end{cases}$$

Let $(\alpha_m)_{m\in\mathbb{N}^\star} \in C_c^\infty(\mathbb{R})$ a sequence of mollifiers, that is :

$$\alpha_m \geq 0 \text{ and } \int_\mathbb{R} \alpha_m = 1$$

and $\Phi_m = \Phi * \alpha_m$, we then get by step1

$$\int_\Omega \Phi_m(\rho)\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \le 0$$

and then using the fact that

$$\Phi_m \to \Phi \text{ a.e in } \mathbb{R}$$

and

$$\|\Phi_m\|_{L^\infty} \le \|\Phi\|_{L^\infty}\|\alpha_m\|_{L^1} = \|\Phi\|_{L^\infty}$$

we thus get applying the Dominated Convergence Theorem

$$\int_\Omega \Phi(\rho)\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \le 0$$

Step 3 In this step we take Φ satisfying

$$\begin{cases} \Phi \in C^0(\mathbb{R}), \text{nondecreasing} \\ \Phi = 0 \text{ on }]-\infty, 0] \\ \Phi = \alpha \text{ on } [a, +\infty[\end{cases}$$

Let $\eta > 0$ and Φ_η defined by $\Phi_\eta(x) = \Phi(x - \eta)$ so that $\Phi_\eta = 0 \text{ on }]-\infty, \eta]$

and then by step2 we get

$$\int_\Omega \Phi_\eta(\rho)\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \le 0$$

Finally, applying the Dominated Convergence Theorem, we get as $\eta \to +\infty$

$$\int_\Omega \Phi(\rho)\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \le 0 \tag{4.4.11}$$

Step 4 In this step we take Φ satisfying

$$\begin{cases} \Phi \in C^0(\mathbb{R}), \text{nondecreasing} \\ \Phi(0) = 0 \\ \Phi(\rho) \in L^2(\Omega) \end{cases}$$

Let $n \in \mathbb{N}^*$ we define Φ_n by $\Phi_n(x) = \min(n, \Phi(x^+))$ where $x^+ = \max(0, x)$

Then, by step3 we have

$$\int_\Omega \Phi_n(\rho)\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \le 0$$

And since $\rho > 0$ and $\Phi(\rho) \in L^2(\Omega)$, applying the Dominated Convergence Theorem we pass to the limit as $n \to +\infty$, and we get

$$\int_\Omega \Phi(\rho)\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \le 0.$$

Remark 4.4.2 *In this step the solution $(\boldsymbol{u}, p, \rho)$ of problem(4.2.5) depends on k, n and m, for simplicity we denote it $(\boldsymbol{u}_m, p_m, \rho_m)$ (since k, n are fixed).*

Proposition 4.4.3 *Let $(\boldsymbol{u}_m, p_m, \rho_m) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}^2(\Omega) \times H^1(\Omega)$ a solution of (4.2.5), then there exists C only depending on k, n and the data of the problem Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, M such that :*

$$\|\boldsymbol{u}_m\|_{H^1(\Omega)^d},\ \|\rho_m\|_{H^1(\Omega)} \ \text{and}\ \|p_m\|_{L^2(\Omega)} \leq C \tag{4.4.12}$$

And then, up to a subsequence, as $m \to +\infty$ we have :

1. *$(\boldsymbol{u}_m)_m$ converges to $\boldsymbol{u}_{k,n}$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)^d$,*
2. *$(\rho_m)_m$ converges to $\rho_{k,n}$ in $L^2(\Omega)$ and weakly in $H^1(\Omega)$,*
3. *$(p_m)_m$ weakly converges to $p_{k,n}$ in $L^2(\Omega)$.*

Proof Let $(\boldsymbol{u}_m, p_m, \rho_m)$ be a solution of (4.2.5) .Taking $\boldsymbol{u}_m$ as test function in (4.2.5a) yields :

$$\|\boldsymbol{u}_m\|^2_{H^1(\Omega)^d} - \int_\Omega p_m \operatorname{div}(\boldsymbol{u}_m)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_m\,\mathrm{d}\boldsymbol{x} + \int_\Omega T_k(\rho_m)\boldsymbol{g}\cdot\boldsymbol{u}_m\,\mathrm{d}\boldsymbol{x}.$$

$$\Rightarrow \|\boldsymbol{u}_m\|^2_{H^1(\Omega)^d} - \int_\Omega \varphi_m(\rho) \operatorname{div}(\boldsymbol{u}_m)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_m\,\mathrm{d}\boldsymbol{x} + \int_\Omega T_k(\rho_m)\boldsymbol{g}\cdot\boldsymbol{u}_m\,\mathrm{d}\boldsymbol{x}.$$

By lemma 4.4.1, we get

$$\|\boldsymbol{u}_m\|^2_{H^1(\Omega)^d} \leq \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_m\,\mathrm{d}\boldsymbol{x} + \int_\Omega T_k(\rho_m)\boldsymbol{g}\cdot\boldsymbol{u}_m\,\mathrm{d}\boldsymbol{x}.$$

Poincaré and Hölder inequalities, yield the existence of C only depending on $\Omega, \boldsymbol{f}, \boldsymbol{g}$ and k such that

$$\|\boldsymbol{u}_m\|_{H^1(\Omega)^d} \leq C. \tag{4.4.13}$$

Using (4.4.13) and theorem(4.3.1) we get :$\exists C$ depending on $\Omega, \boldsymbol{f}, \boldsymbol{g}, k$ and n such that :

$$\|\rho_m\|_{H^1(\Omega)} \leq C$$

In order to obtain a bound for p_m in $\mathrm{L}^2(\Omega)$, we now choose $\boldsymbol{v}$ given by Lemma 3.8.2 with $q = p_m - m(p_m)$, where $m(p_m)$ is the mean value of p_m. Taking $\boldsymbol{v}$ in (4.4.2) and using $\int_\Omega \operatorname{div}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = 0$ gives :

$$\int_\Omega \big(p_m - m(p_m)\big)^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega (\boldsymbol{f}\cdot\boldsymbol{v} + T_k(\rho_m)\boldsymbol{g}\cdot\boldsymbol{v} - \boldsymbol{\nabla}\boldsymbol{u}_m : \boldsymbol{\nabla}\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}.$$

Since $\|\boldsymbol{v}\|_{\mathrm{H}^1(\Omega)^d} \leq c_2\,\|p_m - m(p_m)\|_{\mathrm{L}^2(\Omega)}$ and $\|\boldsymbol{u}_m\|_{\mathrm{H}^1(\Omega)^d} \leq C$, the preceding inequality leads to an estimate on $\|p_m - m(p_m)\|_{\mathrm{L}^2(\Omega)}$, *i.e.* the existence of c_3, only depending on Ω, $\boldsymbol{f}$, $\boldsymbol{g}$ and k , such that $\|p_n - m(p_n)\|_{\mathrm{L}^2(\Omega)} \leq c_3$. We now use the fact that $\int_\Omega \rho_m\,\mathrm{d}\boldsymbol{x} = M$ to deduce an estimate on $\|p_m\|_{L^2(\Omega)}$.

We first modify a little bit the function φ (which is only nondecreasing) in order to obtain a function $\bar\varphi$ continuous and one-to-one from $\mathbb{R}^+$ onto $\mathbb{R}^+$.To construct $\bar\varphi$, we take a continuous increasing function $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\chi(s) \to +\infty$ as $s \to +\infty$ and we set $\bar{\varphi} = (\varphi + \chi)^{-1}$, we thus get

$$\int_\Omega \bar{\varphi}(p_m)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \bar{\varphi}(\varphi_m(\rho_m))\,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \bar{\varphi}((\varphi+\chi)(\rho_m))\ \mathrm{d}\boldsymbol{x} = \int_\Omega \rho_m\ \mathrm{d}\boldsymbol{x} = M.$$

and then, using Lemma 4.4.13 , there exists C, only depending on the Ω, $\boldsymbol{f}$, $\boldsymbol{g}, k$ and M, such that :

$$\|p_m\|_{\mathrm{L}^2(\Omega)} \ \leq C. \tag{4.4.14}$$

The convergence (up to the extraction of a subsequence) of the sequence $(\boldsymbol{u}_m, p_m, \rho_m)$ is a consequence of the uniform (with respect to m) estimates (4.4.12).This concludes the proof.

Proposition 4.4.4 *Let* $(\boldsymbol{u}_{k,n}, p_{k,n}, \rho_{k,n})$ *be the limit of* $(\boldsymbol{u}_m, p_m, \rho_m)$ *in the sense of proposition(4.4.3) ,then*

$(\boldsymbol{u}_{k,n}, p_{k,n}, \rho_{k,n}) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}^2(\Omega) \times H^1(\Omega)$ *is a solution of the following problem :*

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \mathrm{div}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega T_k(\rho)\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\ \forall \boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d, \tag{4.4.15a}$$

$$\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x} - \frac{1}{n}\int_\Omega \boldsymbol{\nabla}\rho(x)\cdot\boldsymbol{\nabla}\psi(x)dx = 0 \ \textit{for all}\ \psi \in H^1(\Omega), \tag{4.4.15b}$$

$$\rho \geq 0\ \textit{a.e. in}\ \Omega,\ \int_\Omega \rho\,\mathrm{d}\boldsymbol{x} = M,\ p = \varphi(\rho)\ \textit{a.e. in}\ \Omega. \tag{4.4.15c}$$

Proof The passage to the limit on the two first equations is quite classical (using proposition4.4.3). We just prove that the equation of state is satisfied, we have

$\rho_m \to \rho_{k,n}$ strongly in $L^2(\Omega)$ and then (up to a subsequence) a.e in Ω which yields

$$p_m = \varphi_m(\rho_m) \to \varphi(\rho_{k,n}) \ \text{a.e in}\ \Omega\ \text{(since φ is continuous)} \tag{4.4.16}$$

On the other hand we have $p_m \to p_{k,n}$ weakly in $L^2(\Omega)$

owing to(4.4.16) we have : $p_m \to p_{k,n}$ strongly in $L^q(\Omega)$, $q < 2$,

we thus get : $p_{k,n} = \varphi(\rho_{k,n})$ a.e in Ω.

- **Fixing n and tending k to $+\infty$**

Proposition 4.4.5 *For n fixed, let* $(\boldsymbol{u}_{k,n}, p_{k,n}, \rho_{k,n}) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}^2(\Omega) \times H^1(\Omega)$ *a solution of the following problem :*

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \mathrm{div}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega T_k(\rho)\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,\forall \boldsymbol{v}\in \mathrm{H}_0^1(\Omega)^d, \quad (4.4.17a)$$

$$\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x} - \frac{1}{n}\int_\Omega \boldsymbol{\nabla}\rho(x)\cdot\boldsymbol{\nabla}\psi(x)dx = 0 \textit{ for all } \psi\in H^1(\Omega), \quad (4.4.17b)$$

$$\rho\geq 0 \textit{ a.e. in } \Omega,\ \int_\Omega \rho\,\mathrm{d}\boldsymbol{x} = M,\ p=\varphi(\rho) \textit{ a.e. in } \Omega. \quad (4.4.17c)$$

Then there exists C only depending on n and the data of the problem Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, M such that :

$$\|\boldsymbol{u}_{k,n}\|_{H^1(\Omega)^d},\ \|\rho_{k,n}\|_{H^1(\Omega)},\ \|p_{k,n}\|_{\mathrm{L}^2(\Omega)} \leq C \quad (4.4.18)$$

And then up to a subsequence, as $k\to+\infty$ we have :

1. *$(\boldsymbol{u}_{k,n})_n$ converges to $\boldsymbol{u}_n$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)^d$,*
2. *$(\rho_{k,n})_n$ converges to ρ_n in $L^2(\Omega)$ and weakly in $H^1(\Omega)$,*
3. *$(p_{k,n})_n$ weakly converges to p_n in $L^2(\Omega)$.*

Proof

For the estimates on $\boldsymbol{u}_{k,n}$ and $p_{k,n}$ we will follow the same steps given in [10] and the H^1-estimate on $\rho_{k,n}$ follows from theorem(4.3.1).

Let $(\boldsymbol{u}_{k,n}, p_{k,n}, \rho_{k,n})$ be a solution of (4.4.17) .Taking $\boldsymbol{u}_{k,n}$ as test function in (4.4.17a) yields :

$$\|\boldsymbol{u}_{k,n}\|^2_{H_0^1(\Omega)^d} - \int_\Omega p_{k,n}\,\mathrm{div}(\boldsymbol{u}_{k,n})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_{k,n}\,\mathrm{d}\boldsymbol{x} + \int_\Omega T_k(\rho_{k,n})\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}. \quad (4.4.19)$$

Using Lemma 4.4.1, Poincaré and Hölder Inequalities, one obtains the existence of C_1 only depending on Ω, $\boldsymbol{f}$, $\boldsymbol{g}$ such that

$$\|\boldsymbol{u}_{k,n}\|_{H_0^1(\Omega)^d} \leq C_1(1+\|\rho_{k,n}\|_{L^2(\Omega)}). \quad (4.4.20)$$

Since $p_{k,n}=\varphi(\rho_{k,n})$, using (4.2.1), for all $\varepsilon>0$ there exists C_ε (only depending on ε, φ and Ω) such that :

$$\|\rho_{k,n}\|_{L^2(\Omega)} \leq C_\varepsilon + \varepsilon\|p_{k,n}\|_{L^2(\Omega)}. \quad (4.4.21)$$

Then, with (4.4.19), for all $\varepsilon>0$, there exists $\bar{C}_\varepsilon$, only depending on $\Omega, \boldsymbol{f}$, $\boldsymbol{g}$, φ and ε such that

$$\|\boldsymbol{u}_{k,n}\|_{H_0^1(\Omega)^d} \leq \bar{C}_\varepsilon + \varepsilon\|p_{k,n}\|_{L^2(\Omega)}. \quad (4.4.22)$$

We now use Lemma 3.8.2 .There exists $w_{k,n} \in H_0^1(\Omega)^d$ such that $\text{div}(w_{k,n}) = p_{k,n} - m(p_{k,n})$ a.e. in Ω and $\|w_{k,n}\|_{H^1(\Omega)^d} \leq c_2 \|p - m(p)\|_{L^2(\Omega)}$ where c_2 only depends on Ω. Taking $v = w_{k,n}$ as test function in (4.4.17a) yields :

$$\int_\Omega p_{k,n} \,\text{div}(v)\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_{k,n} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega T_k(\rho_{k,n}) \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}$$

Since $\int_\Omega \text{div}(v)\, \mathrm{d}\boldsymbol{x} = 0$, this gives also

$$\int_\Omega [p_{k,n} - m(p_{k,n})] \,\text{div}(v)\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_{k,n} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega T_k(\rho_{k,n}) \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}$$

and then

$$\int_\Omega [p_{k,n} - m(p_{k,n})]^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_{k,n} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega T_k(\rho_{k,n}) \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}$$

Using lemma 3.8.2 and the inequalities (4.4.22) and (4.4.21) we get for all $\varepsilon > 0$, the existence of D_ε, only depending on Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, φ and ε such that

$$\|p_{k,n} - m(p_{k,n})\|_{L^2(\Omega)} \leq D_\varepsilon + \varepsilon \|p_{k,n}\|_{L^2(\Omega)}.$$

In order to obtain an estimate on $\|p_{k,n}\|_{L^2}$, we now use the fact that $\int_\Omega \rho_{k,n}\, \mathrm{d}\boldsymbol{x} = M$ and $p_{k,n} = \varphi(\rho_{k,n})$. As in the proof of proposition 4.4.3, we can prove that there exists a continuous and one-to-one function $\bar{\varphi}$ from $\mathbb{R}^+$ onto $\mathbb{R}^+$ and $C > 0$ only depending on M and Ω such that

$$\int_\Omega \bar{\varphi}(p_{k,n})\, \mathrm{d}\boldsymbol{x} \leq C$$

with $\bar{\varphi}$ and p satisfying the conditions of lemma 4.4.13, we thus get the existence of $\bar{C}$, only depending on Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, φ and M such that

$$\|p_{k,n}\|_{L^2(\Omega)} \leq \bar{C} \tag{4.4.23}$$

Using (4.4.23) in (4.4.22) we thus get the estimate on $\|u_{k,n}\|_{H^1(\Omega)^d}$. Finally, thanks to theorem 4.3.1, the estimate on $\|\rho_{k,n}\|_{H^1}$ follows.

The convergence of the sequence $(\boldsymbol{u}_{k,n}, p_{k,n}, \rho_{k,n})$ is a consequence of the uniform (with respect to k) estimates (4.4.18).This concludes the proof.

Proposition 4.4.6 *Let $(\boldsymbol{u}_n, p_n, \rho_n) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}^2(\Omega) \times H^1(\Omega)$ be the limit of $(\boldsymbol{u}_{k,n}, p_{k,n}, \rho_{k,n})$ in the sense of proposition(4.4.5) ,then*

$(\boldsymbol{u}_n, p_n, \rho_n)$ is a solution of the following problem :

$$\int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\text{div}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x},\ \forall \boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d, \tag{4.4.24a}$$

$$\int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi \,\mathrm{d}\boldsymbol{x} - \frac{1}{n} \int_\Omega \boldsymbol{\nabla} \rho(x) \cdot \boldsymbol{\nabla} \psi(x) dx = 0 \textit{ for all } \psi \in H^1(\Omega), \tag{4.4.24b}$$

$$\rho \geq 0 \textit{ a.e. in } \Omega,\ \int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M,\ p = \varphi(\rho) \textit{ a.e. in } \Omega. \tag{4.4.24c}$$

Proof We can easily pass to the limit in the two first equations (using proposition4.4.5), we just give some details for the following terms

For the term $\int_\Omega T_k(\rho_{k,n})\boldsymbol{g}\cdot\boldsymbol{v}$, we use the fact that

$\rho_{k,n}$ strongly converges to ρ_n and $\|\rho_{k,n}\|_{L^2(\Omega)} \leq C$ (C independent of k)

we thus get using the Dominated Convergence Theorem

$$\int_\Omega T_k(\rho_{k,n})\boldsymbol{g}\cdot\boldsymbol{v} \to \int_\Omega \rho_n\,\boldsymbol{g}\cdot\boldsymbol{v} \text{ as } k\to+\infty$$

And for the same arguments we can pass to the limit on the equation of state we thus get

$$p_{k,n} = \varphi(\rho_{k,n}) \to \varphi(\rho_n), \text{ a.e in } \Omega \ (\varphi \text{ continuous})$$

and since

$$p_{k,n} \to p_n \text{ weakly in } L^2(\Omega) \text{ and then strongly in } L^q(\Omega),\, q<2$$

we get by uniqueness of the limit

$$p_n = \varphi(\rho_n) \text{ a.e in } \Omega.$$

which concludes the proof.

- **Tending** n **to** $+\infty$

Proposition 4.4.7 *Let* $(\boldsymbol{u}_n, p_n, \rho_n) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2(\Omega)\times H^1(\Omega)$ *a solution of the following problem :*

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\,\mathrm{div}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho\,\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\ \forall \boldsymbol{v}\in \mathrm{H}^1_0(\Omega)^d, \tag{4.4.25a}$$

$$\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x} - \frac{1}{n}\int_\Omega \boldsymbol{\nabla}\rho(x)\cdot\boldsymbol{\nabla}\psi(x)dx = 0 \textit{ for all } \psi\in H^1(\Omega), \tag{4.4.25b}$$

$$\rho \geq 0 \textit{ a.e. in } \Omega,\ \int_\Omega \rho\,\mathrm{d}\boldsymbol{x} = M,\ p=\varphi(\rho) \textit{ a.e. in } \Omega. \tag{4.4.25c}$$

Then there exists C *only depending on the data of the problem* Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, M *such that :*

$$\|\boldsymbol{u}_n\|_{H^1(\Omega)^d},\ \|\rho_n\|_{L^2(\Omega)},\ \|p_n\|_{\mathrm{L}^2(\Omega)} \leq C \tag{4.4.26}$$

and then up to a subsequence, as $n\to+\infty$ *we have :*

1. $(\boldsymbol{u}_n)_n$ *converges to* $\boldsymbol{u}$ *in* $L^2(\Omega)$ *and weakly in* $H^1_0(\Omega)^d$,
2. $(\rho_n)_n$ *weakly converges to* ρ *in* $L^2(\Omega)$,
3. $(p_n)_n$ *weakly converges to* p *in* $L^2(\Omega)$.

Proof For the estimates on $\|\boldsymbol{u}_n\|_{H^1(\Omega)^d}$ and $\|p_n\|_{\mathrm{L}^2(\Omega)}$ the proof is the same to that given in proposition 4.4.5. It remains to get the estimate on $\|\rho_n\|_{L^2}$ which comes from the estimate on $\|p_n\|_{L^2}$ using $p_n = \varphi(\rho_n)$ and (4.2.1).

The convergence of the sequence $(\boldsymbol{u}_n, p_n, \rho_n)$ is a consequence of the uniform (with respect to n) estimates (4.4.26).This concludes the proof.

Theorem 4.4.8 *The limit $(\boldsymbol{u}, p, \rho) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2(\Omega) \times L^2(\Omega)$ is a solution of the following problem :*

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\operatorname{div}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \textit{ for all } \boldsymbol{v} \in \mathrm{H}^1_0(\Omega)^d, \tag{4.4.27a}$$

$$\int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi \,\mathrm{d}\boldsymbol{x} = 0 \textit{ for all } \psi \in W^{1,\infty}(\Omega), \tag{4.4.27b}$$

$$\rho \geq 0 \textit{ a.e. in } \Omega, \ \int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M, \ p = \varphi(\rho) \textit{ a.e. in } \Omega. \tag{4.4.27c}$$

Proof
This result is obtained by passing to the limit on problem(4.4.25) as $n \to +\infty$.
Using the estimates, we can easily pass to the limit in the first equation .
For the second equation, we have :
$\star \int_\Omega \rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla} \psi \,\mathrm{d}\boldsymbol{x} \to \int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi \,\mathrm{d}\boldsymbol{x}$(proposition 4.4.7).
The second term yields

$$\begin{aligned} \frac{1}{n} \int_\Omega \boldsymbol{\nabla} \rho_n \boldsymbol{\nabla} \psi \,\mathrm{d}\boldsymbol{x} &\leq C_\psi \frac{1}{n} (\int_\Omega \frac{|\boldsymbol{\nabla} \rho_n|^2}{\rho_n} \,\mathrm{d}\boldsymbol{x})^{\frac{1}{2}} (\int_\Omega \rho_n \,\mathrm{d}\boldsymbol{x})^{\frac{1}{2}} \\ &\leq C_{\psi,M} \frac{1}{n} (\int_\Omega \frac{|\boldsymbol{\nabla} \rho_n|^2}{\rho_n} \,\mathrm{d}\boldsymbol{x})^{\frac{1}{2}} \end{aligned}$$

and then

$$\frac{1}{n} \int_\Omega \boldsymbol{\nabla} \rho_n \boldsymbol{\nabla} \psi \,\mathrm{d}\boldsymbol{x} \leq C_{\psi,M} \frac{1}{n} (\int_\Omega \frac{|\boldsymbol{\nabla} \rho_n|^2}{\rho_n} \,\mathrm{d}\boldsymbol{x})^{\frac{1}{2}} \tag{4.4.28}$$

On the other hand, let $j > 0$ taking $\psi = \ln(\rho_n + \frac{1}{j}) \in H^1(\Omega)$ in equation(4.4.25b), we get :

$$\int_\Omega \rho_n \boldsymbol{u}_n \frac{\nabla \rho_n}{\rho_n + \frac{1}{j}} \,\mathrm{d}\boldsymbol{x} - \frac{1}{n} \int_\Omega \frac{|\boldsymbol{\nabla} \rho_n|^2}{\rho_n + \frac{1}{j}} \,\mathrm{d}\boldsymbol{x} = 0$$

Using the Dominated Convergence and the Monotone Convergence Theorems, we pass to the limit as $j \to +\infty$, we thus get

$$\int_\Omega \rho_n \operatorname{div} \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} - \frac{1}{n} \int_\Omega |\boldsymbol{\nabla} \rho_n|^2 / \rho_n \,\mathrm{d}\boldsymbol{x} = 0$$

which yields using proposition(4.4.7) :

$$\frac{1}{n}\int_\Omega |\boldsymbol{\nabla}\rho_n|^2/\rho_n \,\mathrm{d}\boldsymbol{x} \le C, (C \text{ independent of } n)$$

.

Using this estimate in (4.4.28), we thus get :

$$\frac{1}{n}\int_\Omega \boldsymbol{\nabla}\rho_n \boldsymbol{\nabla}\psi \,\mathrm{d}\boldsymbol{x} \le C\frac{1}{\sqrt{n}} \to 0 \text{ as } n \to +\infty$$

We thus get $(\boldsymbol{u}, \rho)$ satisfies equation (4.4.27b).

It remains now to prove that the equation of state is satisfied, that is $p = \varphi(\rho)$ a.e in Ω. This proof is composed of two steps

Step 1. Proving $\int_\Omega \rho_n p_n \,\mathrm{d}\boldsymbol{x} \to \int_\Omega \rho p \,\mathrm{d}\boldsymbol{x}$**.**

Since the sequence $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$, Lemma 4.4.11 gives the existence of a bounded sequence $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ in $\mathrm{H}^1(\Omega)^d$ such that $\mathrm{div}(\boldsymbol{v}_n) = \rho_n$ and $\mathrm{curl}(\boldsymbol{v}_n) = 0$. It is possible to assume (up to a subsequence) that $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\mathrm{L}^2(\Omega)^d$ and weakly in $\mathrm{H}^1(\Omega)^d$. Passing to the limit gives $\mathrm{div}(\boldsymbol{v}) = \rho$ and $\mathrm{curl}(\boldsymbol{v}) = 0$.

$\psi \in C_c^\infty(\Omega)$ (so that $\boldsymbol{v}_n\varphi \in \mathrm{H}_0^1(\Omega)^d$). Taking $\boldsymbol{v} = \boldsymbol{v}_n\psi$ in (4.4.25a) leads to :

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{u}_n : \boldsymbol{\nabla}(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega p_n \,\mathrm{div}(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_n\,\boldsymbol{g}\cdot(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x}.$$

We thus get :

$$\begin{aligned}\int_\Omega \mathrm{div}(\boldsymbol{u}_n)\ \mathrm{div}(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x} &+ \int_\Omega \mathrm{curl}(\boldsymbol{u}_n)\cdot\mathrm{curl}(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x}\\ &- \int_\Omega p_n\,\mathrm{div}(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_n\,\boldsymbol{g}\cdot(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

The choice of $\boldsymbol{v}_n$ gives $\mathrm{div}(\boldsymbol{v}_n\psi) = \rho_n\varphi + \boldsymbol{v}_n\cdot\boldsymbol{\nabla}\psi$ and $\mathrm{curl}(\boldsymbol{v}_n\psi) = L(\psi)\,\boldsymbol{v}_n$, where $L(\psi)$ is a matrix with entries involving the first order derivatives of ψ. Then, the preceding equality yields :

$$\begin{aligned}\int_\Omega \big(\mathrm{div}(\boldsymbol{u}_n) - p_n\big)\,\rho_n\,\psi\,\mathrm{d}\boldsymbol{x} &+ \int_\Omega \mathrm{div}(\boldsymbol{u}_n)\,\boldsymbol{v}_n\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x}\\ &+ \int \mathrm{curl}(\boldsymbol{u}_n)\cdot L(\psi)\,\boldsymbol{v}_n\,\mathrm{d}\boldsymbol{x} - \int_\Omega p_n\boldsymbol{v}_n\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_n\,\boldsymbol{g}\cdot(\boldsymbol{v}_n\psi)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Thanks to the weak convergence of $\boldsymbol{u}_n$ in $\mathrm{H}_0^1(\Omega)^d$ to $\boldsymbol{u}$, the weak convergence of p_n, ρ_n in $\mathrm{L}^2(\Omega)$ to p, ρ(respectivly) and the convergence of $\boldsymbol{v}_n$ in $\mathrm{L}^2(\Omega)^d$ to $\boldsymbol{v}$, we obtain :

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \big(\mathrm{div}(\boldsymbol{u}_n) - p_n\big)\,\rho_n\,\psi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x} &- \int_\Omega \mathrm{div}(\boldsymbol{u})\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x}\\ - \int_\Omega \mathrm{curl}(\boldsymbol{u})\cdot L(\psi)\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} &+ \int_\Omega p\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{4.4.29}$$

Using the fact that $(\boldsymbol{u}, p)$ satisfies (4.4.27a), we get :

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla}(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \mathrm{div}(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho\,\boldsymbol{g}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x},$$

or equivalently :

$$\int_\Omega \mathrm{div}(\boldsymbol{u})\ \mathrm{div}(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{curl}(\boldsymbol{u})\cdot\mathrm{curl}(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \mathrm{div}(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x}$$

$$= \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho\,\boldsymbol{g}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x},$$

which gives (using $\mathrm{div}(\boldsymbol{v}) = \rho$ and $\mathrm{curl}(\boldsymbol{v}) = 0$) :

$$\int_\Omega (\mathrm{div}(\boldsymbol{u}) - p)\,\rho\,\psi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{div}(\boldsymbol{u})\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{curl}(\boldsymbol{u})\cdot L(\psi)\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$
$$-\int_\Omega p\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho\,\boldsymbol{g}\cdot(\boldsymbol{v}\psi)\,\mathrm{d}\boldsymbol{x}.$$

Then, with (4.4.29), we obtain :

$$\lim_{n\to\infty}\int_\Omega \big(p_n - \mathrm{div}(\boldsymbol{u}_n)\big)\,\rho_n\,\psi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(p - \mathrm{div}(\boldsymbol{u})\big)\,\rho\,\psi\,\mathrm{d}\boldsymbol{x}. \tag{4.4.30}$$

In (4.4.30), the function ψ is an arbitrary element of $\mathrm{C}_c^\infty(\Omega)$. We are going to prove now that it is possible to take $\psi = 1$ in this relation, thanks to Lemma 4.4.9. To this goal, we need to prove that the sequence $((p_n - \mathrm{div}(\boldsymbol{u}_n))\,\rho_n)_{n\in\mathbb{N}}$ is equi-integrable. Indeed, using (4.2.1) we can easily prove that the sequence $(\rho_n^2)_{n\in\mathbb{N}^*}$ is equi-integrable. And by proposition4.4.26 we have, the sequance $(p_n - \mathrm{div}(\boldsymbol{u}_n))_{n\in\mathbb{N}^*}$ is bounded in $L^2(\Omega)$, which gives the result

$$\lim_{n\to\infty}\int_\Omega \big(p_n - \mathrm{div}(\boldsymbol{u}_n)\big)\,\rho_n\,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(p - \mathrm{div}(\boldsymbol{u})\big)\,\rho\,\mathrm{d}\boldsymbol{x}. \tag{4.4.31}$$

In order to conclude Step 3, we use Lemma 4.4.1 and 3.7.1, we thus get :

$$\int_\Omega \rho_n\,\mathrm{div}(\boldsymbol{u}_n)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\,\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = 0.$$

Then, (4.4.31) yields :

$$\lim_{n\to\infty}\int_\Omega p_n\,\rho_n\,\mathrm{d}\boldsymbol{x} = \int_\Omega p\,\rho\,\mathrm{d}\boldsymbol{x}. \tag{4.4.32}$$

Step 2. Passing to the limit on the E.O.S.

To conclude the proof of $p = \varphi(\rho)$, we will now use the so called Minty trick. Let $\bar\rho \in L^2(\Omega)$ such that $\varphi(\bar\rho) \in L^2(\Omega)$. We define for $n \in \mathbb{N}$ the function G_n by

$$G_n = (\varphi(\rho_n) - \varphi(\bar\rho))(\rho_n - \bar\rho) = (p_n - \varphi(\bar\rho))(\rho_n - \bar\rho).$$

One has $G_n \in L^1(\Omega)$, $G_n \geq 0$ a.e. in Ω (since φ is nondecreasing) and

$$0 \leq \int_\Omega G_n \, dx = \int_\Omega (p_n\rho_n - p_n\bar{\rho} - \varphi(\bar{\rho})\rho_n + \varphi(\bar{\rho})\bar{\rho}) \, \mathrm{d}\boldsymbol{x}. \tag{4.4.33}$$

Using (4.4.32) and the weak convergences of p_n to p and ρ_n to ρ in $L^2(\Omega)$, we obtain :

$$0 \leq \limsup_{n\to\infty} \int_\Omega G_n \, dx \leq \int_\Omega (p - \varphi(\bar{\rho}))(\rho - \bar{\rho}) \, \mathrm{d}\boldsymbol{x}.$$

We have thus proven that for all $\bar{\rho} \in L^2(\Omega)$ such that $\varphi(\bar{\rho}) \in L^2(\Omega)$ one has

$$\int_\Omega (p - \varphi(\bar{\rho}))(\rho - \bar{\rho}) \, \mathrm{d}\boldsymbol{x} \geq 0. \tag{4.4.34}$$

We now have to choose $\bar{\rho}$ conveniently to deduce $p = \varphi(\rho)$ a.e. on Ω from (3.5.11). The idea of the Minty trick is to take $\bar{\rho} = \rho + (1/k)\psi$ with $\psi \in C_c^\infty(\Omega)$, $k \in \mathbb{N}^\star$ and to let k goes to $+\infty$. Unfortunately, $\varphi(\rho + (1/k)\psi)$ is not necessarily in $L^2(\Omega)$. then, such a choice for $\bar{\rho}$ is not possible. We will use here (and only here) the convexity of φ. Since $(\rho_n)_n$ weakly converges in $L^2(\Omega)$ to ρ and since the sequence $(\varphi(\rho_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, we deduce, using the convexity of φ, that $\varphi(\rho) \in L^2(\Omega)$. This is proven in Lemma 4.4.15. This allows us a convenient choice for $\bar{\rho}$.

Let $\psi \in C_c^\infty(\Omega, \mathbb{R})$. For $k, m \in \mathbb{N}^\star$, we set

$$\rho_{k,m} = \rho + \frac{1}{k}\psi 1_{\rho\leq m}.$$

Since $\rho \in L^2(\Omega)$, one has $\rho_{k,m} \in L^2(\Omega)$. Using the fact that φ is nondecreasing (and nonnegative), we have, with $M = \|\psi\|_{L^\infty(\Omega)}$,

$$\varphi(\rho_{k,m}) \leq \varphi(\rho) + \varphi(m + M),$$

so that $\varphi(\rho_{k,m}) \in L^2(\Omega)$ (since $\varphi(\rho) \in L^2(\Omega)$). Then, since $\rho_{k,m}$ and $\varphi(\rho_{k,m})$ belong to $L^2(\Omega)$, we can choose $\bar{\rho} = \rho_{k,m}$ in (3.5.11). We obtain

$$\int_\Omega (p - \varphi(\rho + \frac{1}{k}\psi 1_{\rho\leq m}))\psi 1_{\rho\leq m} \leq 0.$$

Fixing m in $\mathbb{N}^\star$, we use the Dominated Convergence Theorem on the sequence $(g_k)_{k\in\mathbb{N}^\star}$ with $g_k = (p - \varphi(\rho + \frac{1}{k}\psi 1_{\rho\leq m}))\psi 1_{\rho\leq m}$. Indeed, the continuity of φ gives $g_k \to (p - \varphi(\rho))\psi 1_{\rho\leq m}$ a.e. in Ω. Furthermore, since φ is nondecreasing, one has, for all $n \in \mathbb{N}^\star$,

$$|g_k| \leq H = [p + \varphi(\rho) + \varphi(m + M)]|\psi| \text{ a.e. in } \Omega,$$

and $H \in L^1(\Omega)$. Then, the Dominated Convergence Theorem yields

$$\int_\Omega (p - \varphi(\rho))\psi 1_{\rho\leq m} \leq 0.$$

Changing ψ in $-\psi$, we conclude that $\int_\Omega (p - \varphi(\rho))\psi 1_{\rho\leq m} = 0$ for all $\psi \in C_c^\infty(\Omega, \mathbb{R})$.

Once again by the Dominated Convergence Theorem, as m to $+\infty$ we get : $\int_\Omega (p - \varphi(\rho))\psi = 0$ for all $\psi \in C_c^\infty(\Omega)$ This gives $p = \varphi(\rho)$ a.e. in Ω.

Appendix

Lemma 4.4.9 *Let $(F_n)_{n\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ be an equi-integrable sequence, and F be a function of $\mathrm{L}^1(\Omega)$. We assume that :*

$$\lim_{n\to\infty} \int_\Omega F_n \varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega F\varphi \,\mathrm{d}\boldsymbol{x} \text{ for all } \varphi \in \mathrm{C}_c^\infty(\Omega). \tag{4.4.35}$$

Then :

$$\lim_{n\to\infty} \int_\Omega F_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega F \,\mathrm{d}\boldsymbol{x}.$$

proof This result is proven in [6].

Lemma 4.4.10 *Let $q \in \mathrm{L}^2(\Omega)$ such that $\int_\Omega q \,\mathrm{d}\boldsymbol{x} = 0$. Then, there exists $\boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d$ such that $\mathrm{div}(\boldsymbol{v}) = q$ a.e. in Ω and $\|\boldsymbol{v}\|_{\mathrm{H}^1(\Omega)^d} \le c_2 \|q\|_{\mathrm{L}^2(\Omega)}$ where c_2 only depends on Ω.*

proof The proof is given in [1].

Lemma 4.4.11 *Let Ω be a bounded open set of $\mathbb{R}^d$ and $\rho \in \mathrm{L}^2(\Omega)$. Then, there exists $\boldsymbol{v} \in \mathrm{H}^1(\Omega)^d$ such that $\mathrm{div}(\boldsymbol{v}) = \rho$ a.e. in Ω, $\mathrm{curl}(\boldsymbol{v}) = 0$ a.e. in Ω and $\|\boldsymbol{v}\|_{\mathrm{H}^1(\Omega)^d} \le C \|\rho\|_{\mathrm{L}^2(\Omega)}$ where C only depends on Ω.*
Furthermore, if the boundary of Ω is Lipschitz continuous and $\rho \in \mathrm{H}^1(\Omega)$, it is possible to have $\boldsymbol{v} \in \mathrm{H}^2(\Omega)^d$ and $\|\boldsymbol{v}\|_{\mathrm{H}^2(\Omega)^d} \le C \|\rho\|_{\mathrm{H}^1(\Omega)}$ where C only depends on Ω.

proof This result is proven in [6].

We now prove the following result :

Lemma 4.4.12 *Let Ω be a connected bounded open set of $\mathbb{R}^N$ ($N \ge 1$) with a Lipschitz continuous boundary. Let $\omega \subset \Omega$ be a measurable set with positive Lebesgue measure. We define the set W_ω by :*

$$W_\omega = \{u \in W^{1,1}(\Omega) \text{ such that } u = 0 \text{ a.e. in } \omega\}.$$

Then there exist C only depending on Ω and ω such that

$$\|u\|_{L^p(\Omega)} \le C \| \, |\boldsymbol{\nabla} u| \, \|_{L^1(\Omega)} \text{ for all } u \in W_\omega \text{ and for all } 1 \le p \le \frac{N}{N-1}. \tag{4.4.36}$$

Proof of Lemma 4.4.12

Since Ω is bounded, we only have to prove (4.4.36) for $p = 1^\star = N/(N-1)$. With the Sobolev Embedding Theorem, we already know that there exist C_1 only depending on Ω such that $\|u\|_{L^{1^\star}(\Omega)} \le C_1 \|u\|_{W^{1,1}(\Omega)}$ for all $u \in W^{1,1}(\Omega)$. Then we only have to show that on W_ω the $W^{1,1}$-norm of u is equivalent to the L^1-norm of the gradient of u, that is that there exists C_2 only depending on Ω and ω such that

$$\|u\|_{L^1(\Omega)} \le C_2 \| \, |\boldsymbol{\nabla} u| \, \|_{L^1(\Omega)} \text{ for all } u \in W_\omega. \tag{4.4.37}$$

In order to prove the existence of C_2 such that (4.4.37) holds, we argue by contradiction. We assume the existence of a sequence $(u_n)_{n\in\mathbb{N}^\star}$ in W_ω such that

$$\|u_n\|_{L^1(\Omega)} \ \geq n \, \| \, |\boldsymbol{\nabla} u_n| \, \|_{L^1(\Omega)} \ \text{ for all } n \in \mathbb{N}^\star.$$

Replacing u_n by $u_n / \, \|u_n\|_{L^1(\Omega)}$, we can assume that $\|u_n\|_{L^1(\Omega)} \ = 1$. Then, $(u_n)_{n\in\mathbb{N}^\star}$ is bounded in $W^{1,1}(\Omega)$ and it is relatively compact in $L^1(\Omega)$ (by Rellich' Theorem). Therefore, we can assume (up to a subsequence) that $u_n \to u$ in $L^1(\Omega)$ and a.e..

Furthermore, since

$$\| \, |\boldsymbol{\nabla} u_n| \, \|_{L^1(\Omega)} \ \leq \frac{1}{n},$$

one has $\boldsymbol{\nabla} u = 0$ a.e. in Ω and, since Ω is connected, u is a constant function. Then, the fact that $u_n = 0$ a.e. in ω gives that $u = 0$ a.e. in ω. Therefore $u = 0$ a.e. in Ω. But, this is impossible since $u_n \to u$ in $L^1(\Omega)$ and $\|u_n\|_{L^1(\Omega)}$ =1. This concludes the proof of Lemma 4.4.12.

N.B. It is also possible to prove Lemma 4.4.12 using the "mean-value" Sobolev Inequality (or also using the Poincaré-Wirtinger Inequality). Actually, there exists C_s only depending on Ω such that for all $u \in W^{1,1}(\Omega)$ one has, with $m\lambda_N(\Omega) = \int_\Omega u(x)dx$,

$$\|u - m\|_{L^{1^\star}(\Omega)} \ \leq C_s \, \| \, |\boldsymbol{\nabla} u| \, \|_{L^1(\Omega)} \, .$$

Then, for $u \in W_\omega$, since gives

$$\int_{\Omega\setminus\omega} |u-m|^{1^\star} dx + |m|^{1^\star} \lambda_N(\omega) \leq C_s^{1^\star} \, \| \, |\boldsymbol{\nabla} u| \, \|_{L^1(\Omega)} \, {}^{1^\star}.$$

Then, we have $|m| \leq \frac{C_s}{\lambda_N(\omega)^{1/1^\star}} \, \| \, |\boldsymbol{\nabla} u| \, \|_{L^1(\Omega)}$, and we conclude by using

$$\|u\|_{L^{1^\star}(\Omega)} \ \leq \ \|u-m\|_{L^{1^\star}(\Omega)} \ + |m|\lambda_N(\Omega)^{1/1^\star} \leq C_s\Big(1 + \Big(\frac{\lambda_N(\Omega)}{\lambda_N(\omega)}\Big)^{1/1^\star}\Big) \, \| \, |\boldsymbol{\nabla} u| \, \|_{L^1(\Omega)} \, .$$

The following lemmas are proven in chapter 3 .

Lemma 4.4.13 *Let Ω be a bounded open set of $\mathbb{R}^d$ ($d \geq 1$), and $p \in L^2(\Omega)$,$p \geq 0$ a.e. We assume that there exist $a \in \mathbb{R}$ such that*

$$\|p - m\|_{L^2(\Omega)} \leq a$$

where m is the mean value of p. Furthermore, we assume that there exist $A \in \mathbb{R}$ and a continuous function Ψ from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\int_\Omega \Psi(p) \, \mathrm{d}\boldsymbol{x} \leq A$ and $\Psi(s) \xrightarrow[s\to+\infty]{} +\infty$. Then, there exists C only depending on Ω, a, A and Ψ such that $\|p\|_{L^2(\Omega)} \leq C$.

Lemma 4.4.14 *Let Ω be a bounded open set of $\mathbb{R}^d$. Let $\rho \in L^2(\Omega)$, $\rho \geq 0$ a.e. in Ω and $u \in H_0^1(\Omega)^d$. Assume that (ρ, u) satisfies :*

$$\int_\Omega \rho \, \boldsymbol{u} \cdot \boldsymbol{\nabla}\varphi dx = 0 \textit{ for all } \varphi \in W^{1,\infty}(\Omega). \tag{4.4.38}$$

Then,

$$\int_\Omega \rho \, \mathrm{div}(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} = 0. \tag{4.4.39}$$

Lemma 4.4.15 *Let φ be a convex function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative functions of $L^2(\Omega)$ weakly converging in $L^2(\Omega)$ to ρ. We assume that the sequence $(\varphi(\rho_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. Then, $\varphi(\rho) \in L^2(\Omega)$.*

Conclusions et perspectives

Dans ce travail, on s'est intéressé au problème de Stokes compressible avec une loi d'état très générale. On a établi un résultat d'existence pour ce problème par deux approches : une approche par schéma numérique et une approche par régularité visqueuse. En effet, on a proposé dans le chapitre 3 une discrétisation des équations de Stokes qui combine la méthode des éléments finis et la méthode des volumes finis et qui repose sur les espaces Crouzeix-Raviart. Une première difficulté de ce travail était d'établir des estimations sur la solution discrète, en particulier à cause de la présence de la gravité dans le terme source de l'équation de quantité de mouvement. Une deuxième difficulté dans le passage à la limite sur la loi d'état qui est dûe à sa non-linéarité.

Une deuxième approche pour prouver ce même résultat d'existence a fait l'objet du chapitre 4. En effet, on a démontré ce résultat en utilisant une approximation par viscosité et en passant à la limite sur le problème régularisé.

Les perspectives de ce travail sont nombreuses, en particulier le résultat présenté dans le chapitre 3 peut se faire en utilisant un schéma Mac, ce type de maillage étant plus utilisé en pratique. Il sera aussi très intéressant d'étendre ce travail aux équations de Stokes ou Navier-Stokes instationnaires. Un travail est en cours sur le problème de Stokes instationnaire, des difficultés supplémentaires dans ce cas, en particulier pour démontrer les estimations sur la solution discrète. De plus on sera limité pour des raisons techniques à des équations d'état du type $p = \varphi(\rho)$ avec la fonction φ croissante, convexe et satisfaisant la condition suivante : $\liminf_{s\to+\infty} \varphi(s)/s^{3/2} > 0$ (en dimension 3).

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